

Proofs and Derivations of Binomial Models

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This file contains the proofs and derivations of the 11 binomial models identified in the literature and cited in “A Synthesis of Binomial Option Pricing Models.”

The abbreviations for the models are:

CRR: Cox, Ross, Rubinstein

RBJRT: Randleman-Bartter, Jarrow-Rudd, Jarrow-Turnbull

Wil1, Wil2: These derivations include two models of Wilmott

JKY: Jabbour, Kramin, and Young. These derivations cover five models:

JKYABMD1, JKYRB2, JKYABMC2, JKYABMD2c, and JKYABMD3

The models are covered in these articles.

Chriss, Neil, 1996, *Black-Scholes and Beyond: Option Pricing Models*. New York: McGraw-Hill.

Cox, John C., Stephen A. Ross and Mark Rubinstein, 1979, Option pricing: a simplified approach, *Journal of Financial Economics* 7, 229-264.

Jabbour, George M., Marat V. Kramin, and Stephen D. Young, 2001, Two-state option pricing: binomial models revisited. *Journal of Futures Markets* 11, 987-1001.

Jarrow, Robert A. and Andrew Rudd, 1983, *Option Pricing*. Homewood, Illinois: Richard Irwin.

Jarrow, Robert and Stuart Turnbull, 2000, *Derivative Securities*, 2nd ed. Cincinnati: SouthWestern College Publishing.

Rendleman, Richard J., Jr. and Brit J. Bartter, 1979, Two state option pricing, *The Journal of Finance* 34, 1092-1110.

Trigeorgis, Lenos, 1991, A log-transformed binomial numerical analysis method for valuing complex multi-option investments, *Journal of Financial and Quantitative Analysis* 26, 309-326.

Wilmott, Paul, 1998, *Derivatives: The Theory and Practice of Financial Engineering*. West Sussex, U.K.: Wiley.

CRR

The mean and variance specifications for the physical process are

$$\begin{aligned}q \ln u + (1-q) \ln d &= \mu h \\ q(1-q) [\ln(u/d)]^2 &= \sigma^2 h.\end{aligned}$$

Assume $\ln d = -\ln u$. Then

$$\begin{aligned}q \ln u + (1-q)(-\ln u) &= \mu h \\ 2q \ln u &= \mu h + \ln u \\ q &= \frac{\mu h + \ln u}{2 \ln u} = \frac{\mu h}{2 \ln u} + \frac{\ln u}{2 \ln u} \\ &= \frac{1}{2} + \frac{1}{2} \frac{\mu h}{\ln u}.\end{aligned}$$

Now find $q(1-q)$:

$$\begin{aligned}q(1-q) &= \left(\frac{1}{2} + \frac{1}{2} \frac{\mu h}{\ln u} \right) \left(1 - \left(\frac{1}{2} + \frac{1}{2} \frac{\mu h}{\ln u} \right) \right) = \left(\frac{1}{2} \frac{\mu h}{\ln u} + \frac{1}{2} \right) \left(1 - \frac{1}{2} \frac{\mu h}{\ln u} - \frac{1}{2} \right) \\ &= \left(\frac{1}{2} \frac{\mu h}{\ln u} + \frac{1}{2} \right) \left(\frac{1}{2} - \frac{1}{2} \frac{\mu h}{\ln u} \right) = \frac{1}{4} \frac{\mu h}{\ln u} - \frac{1}{4} \frac{\mu^2 h^2}{(\ln u)^2} + \frac{1}{4} - \frac{1}{4} \frac{\mu h}{\ln u} \\ &= -\frac{1}{4} \frac{\mu^2 h^2}{(\ln u)^2} + \frac{1}{4}.\end{aligned}$$

Then $\ln(u/d) = \ln u - \ln d = \ln u - (-\ln u) = 2 \ln u$. Then

$$\begin{aligned}\sigma^2 h &= q(1-q) [\ln(u/d)]^2 = q(1-q) [2 \ln u]^2 \\ &= q(1-q) 4 (\ln u)^2 = \left(\frac{1}{4} - \frac{1}{4} \frac{\mu^2 h^2}{(\ln u)^2} \right) 4 (\ln u)^2.\end{aligned}$$

CRR assume $h^2 \rightarrow 0$, so

$$\begin{aligned}\left(\frac{1}{4} \right) 4 (\ln u)^2 &= \sigma^2 h \\ \ln u &= \sigma \sqrt{h} \\ u &= e^{\sigma \sqrt{h}}, \quad d = e^{-\sigma \sqrt{h}}.\end{aligned}$$

Working with q :

$$\begin{aligned}q &= \frac{1}{2} + \frac{1}{2} \frac{\mu h}{\ln u} = \frac{1}{2} + \frac{1}{2} \frac{\mu h}{\sigma \sqrt{h}} \\ &= \frac{1}{2} + \frac{1}{2} \frac{\mu}{\sigma} \sqrt{h}.\end{aligned}$$

The limit of q is

$$\lim_{h \rightarrow 0} q = \lim_{h \rightarrow 0} \left(\frac{1}{2} + \frac{1}{2} \frac{\mu}{\sigma} \sqrt{h} \right) = \frac{1}{2}.$$

CRR specify the risk neutral probability to be consistent with no arbitrage:

$$\pi = \frac{e^{rh} - d}{u - d}.$$

In the limit,

$$\lim_{h \rightarrow 0} \pi = \lim_{h \rightarrow 0} \frac{e^{rh} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} = \frac{1-1}{1-1}.$$

Thus, we use L'Hôpital's rule:

$$\begin{aligned} \pi &= \frac{e^{rh} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} = \frac{f(h)}{g(h)} \\ \lim_{h \rightarrow 0} \pi &= \lim_{h \rightarrow 0} \left(\frac{f'(h)}{g'(h)} \right) = \lim_{h \rightarrow 0} \left(\frac{re^{rh} + \frac{\sigma e^{-\sigma\sqrt{h}}}{2\sqrt{h}}}{\frac{\sigma e^{\sigma\sqrt{h}}}{2\sqrt{h}} + \frac{\sigma e^{-\sigma\sqrt{h}}}{2\sqrt{h}}} \right) = \lim_{h \rightarrow 0} \left(\frac{\frac{re^{rh}\sqrt{h}}{\sqrt{h}} + \frac{\sigma e^{-\sigma\sqrt{h}}}{2\sqrt{h}}}{\frac{\sigma e^{\sigma\sqrt{h}}}{\sqrt{h}} + \frac{2\sigma e^{-\sigma\sqrt{h}}}{2\sqrt{h}}} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{re^{rh-\sigma\sqrt{h}}\sqrt{h}}{\sigma} + \frac{1}{2}e^{-2\sigma\sqrt{h}} \right) = 0 + \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

Anomalies:

(1a) Is $\pi < 1$? This will be true if $e^{rh} < u$.

$$\begin{aligned} e^{rh} &< u \\ e^{rh} &< e^{\sigma\sqrt{h}} \\ rh &< \sigma\sqrt{h} \\ r\sqrt{h} &< \sigma \\ h &< \left(\frac{\sigma}{r} \right)^2. \end{aligned}$$

This will not always be true. Thus, π can exceed 1.

(1b) Is $\pi > 0$? This will be true if $e^{rh} > d$. This statement will be true if $d < 1$. The condition of $d < 1$ is evaluated in (3) below and shown to be true. Thus, π always exceeds 0.

(2) Is $u > 1$?

$$u = e^{\sigma\sqrt{h}} > 1.$$

Clearly u is always greater than 1.

(3) Is $d < 1$?

$$d = e^{-\sigma\sqrt{h}} < 1.$$

Thus d is always less than 1.

RBJRT

The mean and variance specifications for the physical process are

$$\begin{aligned} q \ln u + (1-q) \ln d &= \mu h \\ q(1-q) [\ln(u/d)]^2 &= \sigma^2 h. \end{aligned}$$

Set $q = 1/2$. Then, using the variance specification

$$\begin{aligned} \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) [\ln(u/d)]^2 &= \sigma^2 h \\ \ln(u/d) &= 2\sigma\sqrt{h}. \end{aligned}$$

Write the mean specification as

$$q(\ln(u/d)) + \ln d = \mu h.$$

Substituting this result into the mean specification

$$\begin{aligned} \left(\frac{1}{2}\right) (2\sigma\sqrt{h}) + \ln d &= \mu h \\ \ln d &= \mu h - \sigma\sqrt{h} \\ d &= e^{\mu h - \sigma\sqrt{h}}. \end{aligned}$$

Then

$$\begin{aligned} \ln(u/d) &= 2\sigma\sqrt{h} \\ \ln u - \ln d &= 2\sigma\sqrt{h} \\ \ln u &= \ln d + 2\sigma\sqrt{h} \\ &= \mu h + \sigma\sqrt{h} \\ u &= e^{\mu h + \sigma\sqrt{h}}. \end{aligned}$$

To be consistent with no arbitrage, we require $\mu = r - \sigma^2/2$. Therefore,

$$u = e^{(r-\sigma^2/2)h + \sigma\sqrt{h}}, \quad d = e^{(r-\sigma^2/2)h - \sigma\sqrt{h}}.$$

Then JT specify the risk neutral probability to be consistent with no arbitrage:

$$\pi = \frac{e^{rh} - d}{u - d}.$$

Using $\mu = r - \sigma^2/2$ and substituting for u and d :

$$\pi = \frac{e^{rh} - e^{(r-\sigma^2/2)h - \sigma\sqrt{h}}}{e^{(r-\sigma^2/2)h + \sigma\sqrt{h}} - e^{(r-\sigma^2/2)h - \sigma\sqrt{h}}} = \frac{e^{rh} (1 - e^{-\sigma^2 h/2 - \sigma\sqrt{h}})}{e^{rh} (e^{-\sigma^2 h/2 + \sigma\sqrt{h}} - e^{-\sigma^2 h/2 - \sigma\sqrt{h}})}.$$

Multiply by $e^{\sigma^2 h/2} / e^{\sigma^2 h/2}$:

$$\pi = \frac{e^{\sigma^2 h/2} (1 - e^{-\sigma^2 h/2 - \sigma\sqrt{h}})}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}.$$

In the limit,

$$\lim_{h \rightarrow 0} \pi = \lim_{h \rightarrow 0} \frac{e^{\sigma^2 h/2} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} = \frac{1-1}{1-1}.$$

Using L'Hôpital's rule. Let

$$\pi = \frac{e^{\sigma^2 h/2} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} = \frac{f(h)}{g(h)}.$$

Therefore,

$$\begin{aligned} \lim_{h \rightarrow 0} \pi &= \lim_{h \rightarrow 0} \left(\frac{f'(h)}{g'(h)} \right) = \lim_{h \rightarrow 0} \left(\frac{\frac{\sigma^2}{2} e^{\sigma^2 h/2} + \frac{\sigma e^{-\sigma\sqrt{h}}}{2\sqrt{h}}}{\frac{\sigma e^{\sigma\sqrt{h}}}{2\sqrt{h}} + \frac{\sigma e^{\sigma\sqrt{h}}}{2\sqrt{h}}} \right) = \lim_{h \rightarrow 0} \left(\frac{\frac{(1/2)\sqrt{h} e^{\sigma^2 h/2} \sigma^2}{\sqrt{h}}}{\frac{\sigma e^{\sigma\sqrt{h}}}{\sqrt{h}}} + \frac{\frac{(1/2)\sigma e^{-\sigma\sqrt{h}}}{\sqrt{h}}}{\frac{\sigma e^{\sigma\sqrt{h}}}{\sqrt{h}}} \right) \\ &= \lim_{h \rightarrow 0} \left((1/2)\sigma\sqrt{h} e^{\sigma^2 h/2 - \sigma\sqrt{h}} + \frac{1}{2} e^{-2\sigma\sqrt{h}} \right) = 0 + \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

Anomalies:

(1a) Is $\pi < 1$?

This statement will be true if $e^{rh} < u$.

$$\begin{aligned} e^{rh} &< u \\ e^{rh} &< e^{(r-\sigma^2/2)h + \sigma\sqrt{h}} \\ e^{rh} &< e^{rh} \left(e^{-\sigma^2 h/2 + \sigma\sqrt{h}} \right) \\ 1 &< e^{-\sigma^2 h/2 + \sigma\sqrt{h}} \\ e^{-\sigma^2 h/2} &< e^{\sigma\sqrt{h}} \\ \sigma^2 h/2 &< \sigma\sqrt{h} \\ \sigma^2 h &< 2\sigma\sqrt{h} \\ \sigma\sqrt{h} &< 2 \\ \sqrt{h} &< \frac{\sigma}{2} \\ h &< \frac{4}{\sigma^2}. \end{aligned}$$

Thus, π can exceed 1 if h is large enough relative to σ .

(1b) Is $\pi > 0$?

This statement will be true if $e^{rh} > d$.

$$\begin{aligned} e^{rh} &> d \\ e^{rh} &> e^{(r-\sigma^2/2)h - \sigma\sqrt{h}} \\ 1 &> e^{-\sigma^2 h/2 - \sigma\sqrt{h}}. \end{aligned}$$

Thus, π always exceeds 0.

(2) Is $u > 1$?

$$\begin{aligned}e^{(r-\sigma^2/2)h+\sigma\sqrt{h}} &> 1 \\rh - \sigma^2 h/2 + \sigma\sqrt{h} &> 0 \\rh &> \sigma^2 h/2 - \sigma\sqrt{h}.\end{aligned}$$

Thus, u can be less than 1.

(3) Is $d < 1$?

$$\begin{aligned}e^{(r-\sigma^2/2)h-\sigma\sqrt{h}} &< 1 \\rh - \sigma^2 h/2 - \sigma\sqrt{h} &< 0 \\rh &< \sigma^2 h/2 + \sigma\sqrt{h}\end{aligned}$$

Thus, d can exceed 1.

Chriss

The mean and variance specifications for the physical process are

$$\begin{aligned}qu + (1 - q)d &= e^{\alpha h} \\ q(1 - q)[\ln(u/d)]^2 &= \sigma^2 h.\end{aligned}$$

Set $q = 1/2$. Then, the equations become

$$\begin{aligned}\frac{1}{2}u + \frac{1}{2}d &= e^{\alpha h} \\ u + d &= 2e^{\alpha h}.\end{aligned}$$

Using the volatility specification:

$$\begin{aligned}\frac{1}{2} \frac{1}{2} [\ln(u/d)]^2 &= \sigma^2 h \\ u &= de^{2\sigma\sqrt{h}}.\end{aligned}$$

Substituting back:

$$\begin{aligned}de^{2\sigma\sqrt{h}} + d &= 2e^{\alpha h} \\ d(e^{2\sigma\sqrt{h}} + 1) &= 2e^{\alpha h} \\ d &= \frac{2e^{\alpha h}}{e^{2\sigma\sqrt{h}} + 1} \\ u + \frac{2e^{\alpha h}}{e^{2\sigma\sqrt{h}} + 1} &= 2e^{\alpha h} \\ u &= 2e^{\alpha h} - \frac{2e^{\alpha h}}{e^{2\sigma\sqrt{h}} + 1} \\ &= \frac{(e^{2\sigma\sqrt{h}} + 1)2e^{\alpha h} - 2e^{\alpha h}}{e^{2\sigma\sqrt{h}} + 1} \\ u &= \frac{2e^{\alpha h + 2\sigma\sqrt{h}}}{e^{2\sigma\sqrt{h}} + 1}.\end{aligned}$$

To be consistent with no arbitrage, we require $\alpha = r$. Therefore,

$$u = \frac{2e^{rh + 2\sigma\sqrt{h}}}{e^{2\sigma\sqrt{h}} + 1}, \quad d = \frac{2e^{rh}}{e^{2\sigma\sqrt{h}} + 1}.$$

Chriss specifies the risk neutral probability be consistent with no arbitrage:

$$\pi = \frac{e^{rh} - d}{u - d}.$$

Using $\alpha = r$ and substituting for u and d :

$$\begin{aligned}\pi &= \frac{e^{rh} - 2e^{rh} / (e^{2\sigma\sqrt{h}} + 1)}{\frac{2e^{rh+\sigma\sqrt{h}}}{e^{2\sigma\sqrt{h}} + 1} - \frac{2e^{rh}}{e^{2\sigma\sqrt{h}} + 1}} = \frac{e^{rh} (e^{2\sigma\sqrt{h}} + 1) - 2e^{rh}}{e^{2\sigma\sqrt{h}} + 1} \\ &= \frac{e^{rh+\sigma\sqrt{h}} + e^{rh} - 2e^{rh}}{2e^{rh+2\sigma\sqrt{h}} - 2e^{rh}} = \frac{e^{rh} (e^{2\sigma\sqrt{h}} - 1)}{e^{rh} (2e^{2\sigma\sqrt{h}} - 2)} = \left(\frac{1}{2}\right) \frac{e^{2\sigma\sqrt{h}} - 1}{e^{2\sigma\sqrt{h}} - 1} = \frac{1}{2}.\end{aligned}$$

Of course, this result is required because Chriss assumes that $\pi = \frac{1}{2}$ but this proofs confirms that it is upheld.

Anomalies:

(1a) Is $\pi < 1$?

With $\pi = \frac{1}{2}$ there is no question that $\pi < 1$.

(1b) Is $\pi > 0$?

With $\pi = \frac{1}{2}$ there is no question that $\pi < 1$.

(2) Is $u > 1$?

$$\begin{aligned}\frac{2e^{rh+2\sigma\sqrt{h}}}{e^{2\sigma\sqrt{h}} + 1} &> 1 \\ 2e^{rh+2\sigma\sqrt{h}} &> e^{2\sigma\sqrt{h}} + 1 \\ 2e^{rh+2\sigma\sqrt{h}} - e^{2\sigma\sqrt{h}} &> 1 \\ e^{2\sigma\sqrt{h}} (2e^{rh} - 1) &> 1.\end{aligned}$$

The terms in parentheses are at least 1, so u cannot be less than 1.

(3) Is $d < 1$?

$$\begin{aligned}\frac{2e^{rh}}{e^{2\sigma\sqrt{h}} + 1} &< 1 \\ 2e^{rh} &< e^{2\sigma\sqrt{h}} + 1.\end{aligned}$$

Thus, d can exceed 1.

Trigeorgis

The Trigeorgis model transforms the original stock price S to the log of the stock price, X . Let $X = \ln S$. Then the binomial process specifies that X goes to $X + \Delta X$ or $X - \Delta X$ with probabilities q and $1 - q$. Thus, the binomial process has $\ln S$ going to $\ln S^+ = \ln S + \Delta X$ or to $\ln S^- = \ln S - \Delta X$. Thus,

$$\begin{aligned}\ln S^+ &= \ln S + \Delta X \\ S^+ &= e^{\ln S + \Delta X} = S e^{\Delta X} = Su \\ u &= e^{\Delta X} \\ \Delta X &= \ln u \\ \ln S^- &= \ln S - \Delta X \\ S^- &= e^{\ln S - \Delta X} = S e^{-\Delta X} = Sd \\ d &= e^{-\Delta X} \\ -\Delta X &= \ln d.\end{aligned}$$

The expected value of ΔX is

$$\begin{aligned}E(\Delta X) &= q\Delta X + (1-q)(-\Delta X) = q\Delta X - \Delta X + q\Delta X \\ &= 2q\Delta X - \Delta X = \Delta X(2q-1).\end{aligned}$$

The variance is

$$\begin{aligned}\text{Var}(\Delta X) &= q(\Delta X - E(\Delta X))^2 + (1-q)(-\Delta X - E(\Delta X))^2 \\ &= q(\Delta X - (2q\Delta X - \Delta X))^2 + (1-q)(-\Delta X - (2q\Delta X - \Delta X))^2 \\ &= q(\Delta X - 2q\Delta X + \Delta X)^2 + (1-q)(-2q\Delta X)^2 \\ &= q(2\Delta X - 2q\Delta X)^2 + (1-q)(-2q\Delta X)^2 \\ &= q(4(\Delta X)^2 - 8q(\Delta X)^2 + 4q^2(\Delta X)^2) + (1-q)4q^2(\Delta X)^2 \\ &= 4q(\Delta X)^2 - 8q^2(\Delta X)^2 + 4q^3(\Delta X)^2 + 4q^2(\Delta X)^2 - 4q^3(\Delta X)^2 \\ &= 4q(\Delta X)^2 - 4q^2(\Delta X)^2 = 4(\Delta X)^2 q(1-q).\end{aligned}$$

To derive q , use the expected return:

$$\begin{aligned}E(\Delta X) &= 2q\Delta X - \Delta X = \mu h \\ \mu h + \Delta X &= 2q\Delta X \\ q &= \frac{1}{2} \left(\frac{\mu h}{\Delta X} + 1 \right).\end{aligned}$$

Now we need $q(1 - q)$:

$$\begin{aligned}
\frac{1}{2}\left(\frac{\mu h}{\Delta X}+1\right)\left(1-\frac{1}{2}\left(\frac{\mu h}{\Delta X}+1\right)\right) &= \left(\frac{1}{2}\frac{\mu h}{\Delta X}+\frac{1}{2}\right)\left(1-\frac{1}{2}\frac{\mu h}{\Delta X}-\frac{1}{2}\right) \\
&= \left(\frac{1}{2}\frac{\mu h}{\Delta X}+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{2}\frac{\mu h}{\Delta X}\right) \\
&= \frac{1}{4}\frac{\mu h}{\Delta X}-\frac{1}{4}\frac{(\mu h)^2}{4(\Delta X)^2}+\frac{1}{4}-\frac{1}{4}\frac{\mu h}{\Delta X} \\
&= -\frac{1}{4}\frac{(\mu h)^2}{4(\Delta X)^2}+\frac{1}{4}.
\end{aligned}$$

Insert back into the variance expression:

$$\begin{aligned}
4(\Delta X)^2 q(1-q) &= 4(\Delta X)^2 - \left(\frac{1}{4}\frac{(\mu h)^2}{(\Delta X)^2} + \frac{1}{4}\right) \\
&= -4\frac{(\Delta X)^2(\mu h)^2}{4(\Delta X)^2} + 4(\Delta X)^2\frac{1}{4} = -(\mu h)^2 + (\Delta X)^2 \\
\sigma^2 h &= (\Delta X)^2 - (\mu h)^2.
\end{aligned}$$

Now we can solve for ΔX :

$$\begin{aligned}
\sigma^2 h &= (\Delta X)^2 - (\mu h)^2 \\
(\Delta X)^2 &= \sigma^2 h + (\mu h)^2 \\
\Delta X &= \sqrt{\sigma^2 h + (\mu h)^2}.
\end{aligned}$$

The limit of q is easily found as

$$\lim_{h \rightarrow 0} q = \lim_{h \rightarrow 0} \left(\frac{1}{2} \left(\frac{\mu h}{\Delta X} + 1 \right) \right) = \frac{1}{2} \left(\frac{0}{0} + 1 \right) = \frac{1}{2}.$$

We will use L'Hôpital's rule.

$$\begin{aligned}
\lim_{h \rightarrow 0} q &= \lim_{h \rightarrow 0} \left(\frac{1}{2} \frac{\mu h}{\sqrt{\sigma^2 h + (\mu h)^2}} \right) + \frac{1}{2} \\
q &= \frac{f(h)}{g(h)} + \frac{1}{2} \\
\lim_{h \rightarrow 0} q &= \lim_{h \rightarrow 0} \left(\frac{f'(h)}{g'(h)} \right) + \frac{1}{2} \\
f'(h) &= \mu \\
g'(h) &= \frac{\sigma^2 + 2\mu^2 h}{\sqrt{\sigma^2 h + (\mu h)^2}} \\
\lim_{h \rightarrow 0} q &= \lim_{h \rightarrow 0} \left(\frac{\mu}{\frac{\sigma^2 + 2\mu^2 h}{\sqrt{\sigma^2 h + (\mu h)^2}}} \right) + \frac{1}{2} = \lim_{h \rightarrow 0} \left(\frac{\mu \sqrt{\sigma^2 h + (\mu h)^2}}{\sigma^2 + 2\mu^2 h} \right) + \frac{1}{2} = \frac{1}{2}.
\end{aligned}$$

To risk neutralize, we substitute $r - \sigma^2/2$ for μ :

$$\Delta X = \sqrt{\sigma^2 h + (r - \sigma^2 / 2)^2 h^2}$$

$$u = e^{\sqrt{\sigma^2 h + (r - \sigma^2 / 2)^2 h^2}}, \quad d = e^{-\sqrt{\sigma^2 h + (r - \sigma^2 / 2)^2 h^2}}$$

$$\pi^* = \frac{1}{2} \left(\frac{(r - \sigma^2 / 2)h}{\Delta X} + 1 \right).$$

The limit of the formula for π^* is $\frac{1}{2}$ because we have already shown that the limit of q is $\frac{1}{2}$, and π^* is just q with $(r - \sigma^2 / 2)$ substituted for μ . The limit of π would be found as follows:

$$\pi = \frac{e^{rh} - d}{u - d} = \frac{e^{-rh} - e^{-\sqrt{\sigma^2 h + (r - \sigma^2 / 2)^2 h^2}}}{e^{\sqrt{\sigma^2 h + (r - \sigma^2 / 2)^2 h^2}} - e^{-\sqrt{\sigma^2 h + (r - \sigma^2 / 2)^2 h^2}}}$$

$$\lim_{h \rightarrow 0} \pi = \frac{1 - 1}{1 - 1}.$$

So we will use L'Hôpital's rule. In the derivation that follows some simplifications are made, such as using ΔX instead of its detailed formula.

$$\pi = \frac{f(h)}{g(h)}$$

$$\lim_{h \rightarrow 0} \pi = \lim_{h \rightarrow 0} \left(\frac{f'(h)}{g'(h)} \right)$$

$$f(h) = e^{rh} - e^{-\Delta X}, \quad g(h) = e^{\Delta X} - e^{-\Delta X}$$

$$f'(h) = re^{rh} + \frac{1}{2} \frac{e^{\Delta X} (\sigma^2 + 2(r - \sigma^2 / 2)^2 h)}{\sqrt{\Delta X}}$$

$$g'(h) = \frac{1}{2} \frac{e^{\Delta X} (\sigma^2 + 2(r - \sigma^2 / 2)^2 h)}{\sqrt{\Delta X}} + \frac{1}{2} \frac{e^{\Delta X} (\sigma^2 + 2(r - \sigma^2 / 2)^2 h)}{\sqrt{\Delta X}} = \frac{e^{\Delta X} (\sigma^2 + 2(r - \sigma^2 / 2)^2 h)}{\sqrt{\Delta X}}$$

$$\frac{f'(h)}{g'(h)} = \frac{\frac{re^{rh} \Delta X + (1/2)e^{\Delta X} (\sigma^2 + 2(r - \sigma^2 / 2)^2 h)}{\sqrt{\Delta X}}}{\frac{e^{\Delta X} (\sigma^2 + 2(r - \sigma^2 / 2)^2 h)}{\sqrt{\Delta X}}}$$

$$= \frac{re^{rh} \Delta X + (1/2)e^{\Delta X} (\sigma^2 + 2(r - \sigma^2 / 2)^2 h)}{e^{\Delta X} (\sigma^2 + 2(r - \sigma^2 / 2)^2 h)}$$

$$\lim_{h \rightarrow 0} \left(\frac{f'(h)}{g'(h)} \right) = \frac{0 + (1/2)(1)(\sigma^2 + 0)}{1(\sigma^2 + 0)} = \frac{1}{2}.$$

Anomalies:

(1a) Is $\pi^* < 1$?

$$\begin{aligned}
& \pi' < 1 \\
& \frac{1}{2} \left(\frac{(r - \sigma^2/2)h}{\sqrt{\sigma^2 h + (r - \sigma^2/2)^2 h^2}} + 1 \right) < 1 \\
& \frac{(r - \sigma^2/2)h}{\sqrt{\sigma^2 h + (r - \sigma^2/2)^2 h^2}} + 1 < 2 \\
& \frac{(r - \sigma^2/2)h}{\sqrt{\sigma^2 h + (r - \sigma^2/2)^2 h^2}} < 1 \\
& (r - \sigma^2/2)h < \sqrt{\sigma^2 h + (r - \sigma^2/2)^2 h^2} \\
& (r - \sigma^2/2)^2 h^2 < \sigma^2 h + (r - \sigma^2/2)^2 h^2 \\
& 0 < \sigma^2 h.
\end{aligned}$$

Thus, π is always less than 1.

(1b) Is $\pi^* > 0$?

$$\begin{aligned}
& \pi^* > 0 \\
& \frac{1}{2} \left(\frac{(r - \sigma^2/2)h}{\sqrt{\sigma^2 h + (r - \sigma^2/2)^2 h^2}} + 1 \right) > 0 \\
& \frac{(r - \sigma^2/2)h}{\sqrt{\sigma^2 h + (r - \sigma^2/2)^2 h^2}} + 1 > 0 \\
& \frac{(r - \sigma^2/2)h}{\sqrt{\sigma^2 h + (r - \sigma^2/2)^2 h^2}} > -1 \\
& (r - \sigma^2/2)h > -\sqrt{\sigma^2 h + (r - \sigma^2/2)^2 h^2} \\
& (r - \sigma^2/2)^2 h^2 < \sigma^2 h + (r - \sigma^2/2)^2 h^2 \\
& 0 < \sigma^2 h.
\end{aligned}$$

Thus, π^* is always greater than 0.

(2) Is $u > 1$?

$$e^{\Delta X} > 1.$$

Because $\Delta X > 0$, u is always greater than 1.

(3) Is $d < 1$?

$$e^{-\Delta X} < 1.$$

Because $\Delta X > 0$, d is always less than 1.

Wilmott1 and 2

The mean and variance specifications for the physical process are

$$\begin{aligned}qu + (1 - q)d &= e^{\alpha h} \\ q(u - e^{\alpha h})^2 + (1 - q)(d - e^{\alpha h})^2 &= e^{2\alpha h}(e^{\sigma^2 h} - 1).\end{aligned}$$

Working with the variance:

$$\begin{aligned}q(u - e^{\alpha h})^2 + (1 - q)(d - e^{\alpha h})^2 &= qu^2 + (1 - q)d^2 - (e^{\alpha h})^2 \\ qu^2 + (1 - q)d^2 - (e^{\alpha h})^2 &= e^{2\alpha h}(e^{\sigma^2 h} - 1) \\ qu^2 + (1 - q)d^2 &= e^{(2\alpha + \sigma^2)h} \\ qu^2 + d^2 - qd^2 &= e^{(2\alpha + \sigma^2)h} \\ q(u^2 - d^2) &= e^{(2\alpha + \sigma^2)h} - d^2 \\ q &= \frac{e^{(2\alpha + \sigma^2)h} - d^2}{u^2 - d^2}.\end{aligned}$$

Working with the mean:

$$\begin{aligned}qu + d - qd &= e^{\alpha h} \\ q &= \frac{e^{\alpha h} - d}{u - d}.\end{aligned}$$

Setting these equations equal to each other:

$$\begin{aligned}\frac{e^{(2\alpha + \sigma^2)h} - d^2}{u^2 - d^2} &= \frac{e^{\alpha h} - d}{u - d} \\ \frac{e^{(2\alpha + \sigma^2)h} - d^2}{(u - d)(u + d)} &= \frac{e^{\alpha h} - d}{u - d} \\ (e^{(2\alpha + \sigma^2)h} - d^2)(u - d) &= (e^{\alpha h} - d)(u - d)(u + d) \\ u + d &= \frac{e^{(2\alpha + \sigma^2)h} - d^2}{e^{\alpha h} - d}.\end{aligned}$$

Will:

For the special case of Will, let $u = 1/d$.

$$\begin{aligned}\frac{1}{d} + d &= \frac{e^{(2\alpha + \sigma^2)h} - d^2}{e^{\alpha h} - d} \\ \frac{1 + d^2}{d} &= \frac{e^{(2\alpha + \sigma^2)h} - d^2}{e^{\alpha h} - d} \\ (1 + d^2)(e^{\alpha h} - d) &= (e^{(2\alpha + \sigma^2)h} - d^2)d \\ e^{\alpha h} - d + d^2 e^{\alpha h} - d^3 &= e^{(2\alpha + \sigma^2)h} d - d^3 \\ d^2 e^{\alpha h} - d - d e^{(2\alpha + \sigma^2)h} + e^{\alpha h} &= 0.\end{aligned}$$

Now multiply by $e^{-\alpha h}$:

$$d^2 - de^{-\alpha h} - de^{(\alpha+\sigma^2)h} + 1 = 0$$

$$d^2 - d(e^{-\alpha h} + e^{(\alpha+\sigma^2)h}) + 1 = 0.$$

Using the quadratic formula:

$$d = \frac{(e^{-\alpha h} + e^{(\alpha+\sigma^2)h}) \pm \sqrt{(e^{-\alpha h} + e^{(\alpha+\sigma^2)h})^2 - 4}}{2} = \frac{1}{2}(e^{-\alpha h} + e^{(\alpha+\sigma^2)h}) - \frac{1}{2}\sqrt{(e^{-\alpha h} + e^{(\alpha+\sigma^2)h})^2 - 4}$$

$$u = \frac{1}{2}(e^{-\alpha h} + e^{(\alpha+\sigma^2)h}) + \frac{1}{2}\sqrt{(e^{-\alpha h} + e^{(\alpha+\sigma^2)h})^2 - 4}.$$

To risk neutralize, let $\alpha = r$:

$$u = \frac{1}{2}(e^{-rh} + e^{(r+\sigma^2)h}) + \frac{1}{2}\sqrt{(e^{-rh} + e^{(r+\sigma^2)h})^2 - 4}$$

$$d = \frac{1}{2}(e^{-rh} + e^{(r+\sigma^2)h}) - \frac{1}{2}\sqrt{(e^{-rh} + e^{(r+\sigma^2)h})^2 - 4}.$$

The risk neutral probability is upheld because the raw mean constraint was used. Thus,

$$\pi = \frac{e^{rh} - d}{u - d}.$$

with u and d as specified directly above.

To determine the limit of π , we substitute for u and d :

$$\pi = \frac{e^{rh} - \left(\frac{1}{2}(e^{-rh} + e^{(r+\sigma^2)h}) - \frac{1}{2}\sqrt{(e^{-rh} + e^{(r+\sigma^2)h})^2 - 4} \right)}{\left(\frac{1}{2}(e^{-rh} + e^{(r+\sigma^2)h}) + \frac{1}{2}\sqrt{(e^{-rh} + e^{(r+\sigma^2)h})^2 - 4} \right) - \left(\frac{1}{2}(e^{-rh} + e^{(r+\sigma^2)h}) - \frac{1}{2}\sqrt{(e^{-rh} + e^{(r+\sigma^2)h})^2 - 4} \right)}$$

$$= \frac{e^{rh} - \frac{1}{2}(e^{-rh} + e^{(r+\sigma^2)h}) + \frac{1}{2}\sqrt{(e^{-rh} + e^{(r+\sigma^2)h})^2 - 4}}{\sqrt{(e^{-rh} + e^{(r+\sigma^2)h})^2 - 4}}.$$

taking the limit gives

$$\frac{1 - (1/2)(1) - (1/2)1 - (1/2)\sqrt{0}}{\sqrt{0}} = \frac{1-1}{0}.$$

So we will need to use L'Hôpital's rule. Define π as follows:

$$\pi = \frac{f(h)}{g(h)} \text{ where}$$

$$f(h) = e^{rh} - \frac{1}{2}(e^{-rh} + e^{(r+\sigma^2)h}) + \frac{1}{2}\sqrt{(e^{-rh} + e^{(r+\sigma^2)h})^2 - 4}$$

$$g(h) = \sqrt{(e^{-rh} + e^{(r+\sigma^2)h})^2 - 4}.$$

Then

$$\lim_{h \rightarrow 0} \pi = \lim_{h \rightarrow 0} \left(\frac{f'(h)}{g'(h)} \right)$$

$$\begin{aligned} f'(h) &= re^{rh} + \frac{1}{2}re^{-rh} - \frac{1}{2}(r + \sigma^2)e^{(r+\sigma^2)h} + \frac{1}{2} \frac{2(e^{-rh} + e^{(r+\sigma^2)h})(-re^{-rh} + (r + \sigma^2)e^{(r+\sigma^2)h})}{g(h)} \\ &= re^{rh} + \frac{1}{2}re^{-rh} - \frac{1}{2}(r + \sigma^2)e^{(r+\sigma^2)h} + \frac{1}{2} \frac{(e^{-rh} + e^{(r+\sigma^2)h})(-re^{-rh} + (r + \sigma^2)e^{(r+\sigma^2)h})}{g(h)} \\ &= \frac{g(h) \left(re^{rh} + \frac{1}{2}re^{-rh} - \frac{1}{2}(r + \sigma^2)e^{(r+\sigma^2)h} \right) + \frac{1}{2}(e^{-rh} + e^{(r+\sigma^2)h})(-re^{-rh} + (r + \sigma^2)e^{(r+\sigma^2)h})}{g(h)} \end{aligned}$$

$$\begin{aligned} g'(h) &= \frac{(-re^{-rh} + (r + \sigma^2)e^{(r+\sigma^2)h})(e^{-rh} + e^{(r+\sigma^2)h})}{g(h)} \\ &= \frac{g(h)(e^{-rh} + e^{(r+\sigma^2)h}) \left(re^{rh} + \frac{1}{2}re^{-x} - \frac{1}{2}(r + \sigma^2)e^{(r+\sigma^2)h} \right) + \frac{1}{2}(e^{-rh} + e^{(r+\sigma^2)h})(-re^{-rh} + (r + \sigma^2)e^{(r+\sigma^2)h})}{g(h)} \\ \frac{f'(h)}{g'(h)} &= \frac{g(h)}{(-re^{-rh} + (r + \sigma^2)e^{(r+\sigma^2)h})(e^{-rh} + e^{(r+\sigma^2)h})} \\ &= \frac{g(h)(e^{-rh} + e^{(r+\sigma^2)h}) \left(re^{rh} + \frac{1}{2}re^{-x} - \frac{1}{2}(r + \sigma^2)e^{(r+\sigma^2)h} \right) + \frac{1}{2}(-re^{-rh} + (r + \sigma^2)e^{(r+\sigma^2)h})(e^{-rh} + e^{(r+\sigma^2)h})}{(-re^{-rh} + (r + \sigma^2)e^{(r+\sigma^2)h})(e^{-rh} + e^{(r+\sigma^2)h})} \end{aligned}$$

$$\lim_{h \rightarrow 0} \frac{f'(h)}{g'(h)} = \frac{0 + \frac{1}{2}(-r + (r + \sigma^2))2}{(-r + (r + \sigma^2))2} = \frac{1}{2}.$$

Anomalies:

(1a) Is $\pi > 0$?

This statement will be true if $u > e^{rh}$.

$$\frac{1}{2}(e^{-rh} + e^{(r+\sigma^2)h}) + \frac{1}{2}\sqrt{(e^{-rh} + e^{(r+\sigma^2)h})^2 - 4} > e^{rh}.$$

Let $x = rh$ and $y = \sigma^2h$.

$$\frac{1}{2}(e^{-x} + e^y) + \frac{1}{2}\sqrt{(e^{-x} + e^{x+y})^2 - 4} > e^x.$$

Make the assumption that $y = 0$. If the LSH exceeds the RHS with $y = 0$, it will also be true if $y > 0$.

$$\begin{aligned} \frac{1}{2}(e^{-x} + e^x) + \frac{1}{2}\sqrt{(e^{-x} + e^x)^2 - 4} &> e^x \\ \frac{1}{2}(e^{-x} + e^x) + \frac{1}{2}\sqrt{e^{-2x} + e^{2x} + 2} - 4 &> e^x \\ \frac{1}{2}(e^{-x} + e^x) + \frac{1}{2}\sqrt{(e^x + e^{-x})^2} &> e^x \\ \frac{1}{2}(e^{-x} + e^x) + \frac{1}{2}(e^x + e^{-x}) &> e^x \\ e^x &= e^x. \end{aligned}$$

We have shown that the LHS equals the RHS with $y = 0$. Thus, the LHS exceeds the RHS with $y > 0$. Therefore, (1a) is true and π is always less than 1.

(1b) Is $\pi > 0$?

This statement will be true if $d < e^{rh}$. Because $e^{rh} > 1$, this statement will be true if $d < 1$, which will be proven in (3). Thus, we will be able to say that π is greater than 0.

(2) Is $u > 1$?

We have already shown that $u > e^{rh}$ and since $e^{rh} > 1$, u is always greater than 1.

(3) Is $d < 1$?

This statement is somewhat complex to prove. Using the notation used above of $x = rh$ and $y = \sigma^2 h$, assume that $x = 0$ and $y = 0$. Then

$$d = \frac{1}{2}(1+1) - \frac{1}{2}\sqrt{(1+1)^2 - 4} = 1.$$

If we can show that $\partial d/\partial x$ and $\partial d/\partial y$ are both negative, then we have shown that $d < 1$. First specify:

$$B = e^{-x} + e^{x+y}.$$

So that

$$d = \frac{1}{2}B - \frac{1}{2}\sqrt{B^2 - 4}.$$

The partial derivatives are:

$$\begin{aligned}\frac{\partial B}{\partial x} &= -e^{-x} + e^{x+y}, & \frac{\partial B^2}{\partial x} &= 2B \frac{\partial B}{\partial x} = 2B(-e^{-x} + e^{x+y}) \\ \frac{\partial \sqrt{B^2 - 4}}{\partial x} &= \frac{1}{2}(B^2 - 4)^{-1/2} 2B(-e^{-x} + e^{x+y}), & \frac{\partial B^2}{\partial x} &= 2B(-e^{-x} + e^{x+y}) \\ \frac{\partial B}{\partial y} &= e^{x+y}, & \frac{\partial (B^2)}{\partial y} &= 2B \frac{\partial B}{\partial y} = 2Be^{x+y} \\ \frac{\partial \sqrt{B^2 - 4}}{\partial y} &= \frac{1}{2}(B^2 - 4)^{-1/2} 2Be^{x+y}, & \frac{\partial B^2}{\partial y} &= 2Be^{x+y}.\end{aligned}$$

Therefore,

$$\frac{\partial d}{\partial x} = \frac{1}{2}(-e^{-x} + e^{x+y}) \left(1 - \frac{B}{\sqrt{B^2 - 4}} \right).$$

The sum of the exponential expressions in parentheses is positive because $e^x > -e^{-x}$ and $e^y > 1$. Now we show:

$$\begin{aligned}\frac{B}{\sqrt{B^2 - 4}} &> 1 \\ B &> \sqrt{B^2 - 4} \\ B^2 &> B^2 - 4 \\ 0 &> -4.\end{aligned}$$

Thus, the term in large parentheses in the formula for $\partial d/\partial x$ is negative and the whole expression is negative. Looking at $\partial d/\partial y$, we have

$$\frac{\partial d}{\partial y} = \frac{1}{2}(e^{x+y}) \left(1 - \frac{B}{\sqrt{B^2 - 4}} \right).$$

By the same arguments, we obtain $\partial d/\partial y < 0$. Thus, d is always less than 1.

Wil2:

Now look at Wilmott2, which assumes that $q = 1/2$. Therefore, we have

$$\frac{e^{\alpha h} - d}{u - d} = \frac{1}{2}.$$

Using a previous result for the general case,

$$\frac{e^{(2\alpha + \sigma^2)h} - d^2}{u^2 - d^2} = \frac{1}{2}.$$

We now have the following:

$$\begin{aligned}\frac{1}{2}u + \frac{1}{2}d &= e^{\alpha h} \\ u + d &= 2e^{\alpha h} \\ \frac{1}{2}u^2 + \frac{1}{2}d^2 &= e^{(2\alpha + \sigma^2)h} \\ u^2 + d^2 &= 2e^{(2\alpha + \sigma^2)h}.\end{aligned}$$

Letting $d = 2e^{\alpha h} - u$,

$$\begin{aligned}
u^2 + (2e^{\alpha h} - u)^2 &= 2e^{(2\alpha + \sigma^2)h} \\
u^2 + (4e^{2\alpha h} - 4e^{\alpha h}u + u^2) &= 2e^{(2\alpha + \sigma^2)h} \\
2u^2 + 4e^{2\alpha h} - 4e^{\alpha h}u &= 2e^{(2\alpha + \sigma^2)h} \\
u^2 + 2e^{2\alpha h} - 2e^{\alpha h}u &= e^{(2\alpha + \sigma^2)h} \\
u^2 + 2e^{2\alpha h} - 2e^{\alpha h}u - e^{(2\alpha + \sigma^2)h} &= 0 \\
u &= \frac{2e^{2\alpha h} \pm \sqrt{4e^{2\alpha h} - 4(2e^{2\alpha h} - e^{(2\alpha + \sigma^2)h})}}{2} \\
&= e^{\alpha h} \pm \frac{1}{2} \sqrt{4e^{2\alpha h} - 8e^{2\alpha h} + 4e^{(2\alpha + \sigma^2)h}} \\
&= e^{\alpha h} \pm \frac{1}{2} \sqrt{4e^{2\alpha h} (1 - 2 + e^{\sigma^2 h})} = e^{\alpha h} \pm \frac{1}{2} 2e^{\alpha h} \sqrt{e^{\sigma^2 h} - 1} \\
u &= e^{\alpha h} \left(1 + \sqrt{e^{\sigma^2 h} - 1} \right) \\
d &= e^{\alpha h} \left(1 - \sqrt{e^{\sigma^2 h} - 1} \right).
\end{aligned}$$

To risk neutralize, we set $\alpha = r$:

$$u = e^{(r - \sigma^2/2)h} \left(1 + \sqrt{e^{\sigma^2 h} - 1} \right), \quad d = e^{(r - \sigma^2/2)h} \left(1 - \sqrt{e^{\sigma^2 h} - 1} \right).$$

Because the raw mean is the constraint, the risk neutral probability is correctly specified:

$$\pi = \frac{e^{rh} - d}{u - d} = \frac{1}{2}.$$

Of course, there is no need to check the limit of π as it is always $\frac{1}{2}$.

Anomalies:

(1a) Is $\pi < 1$?

This statement will be true if $e^{rh} < u$.

$$\begin{aligned}
e^{rh} &< e^{rh} \left(1 + \sqrt{e^{\sigma^2 h} - 1} \right) \\
1 &< 1 + \sqrt{e^{\sigma^2 h} - 1} \\
0 &< \sqrt{e^{\sigma^2 h} - 1}.
\end{aligned}$$

This statement is always true so π is always less than 1.

(1b) Is $\pi > 0$?

This statement will be true if $e^{rh} > d$?

$$\begin{aligned}
e^{rh} &> e^{rh} \left(1 - \sqrt{e^{\sigma^2 h} - 1}\right) \\
1 &> 1 - \sqrt{e^{\sigma^2 h} - 1} \\
\sqrt{e^{\sigma^2 h} - 1} &> 0.
\end{aligned}$$

This statement is obviously true so π is always greater than 0.

(2) Is $u > 1$?

$$e^{rh} \left(1 + \sqrt{e^{\sigma^2 h} - 1}\right) > 1.$$

Since $e^{rh} > 1$ and we have already shown that the term in parentheses is greater than zero, then u is always greater than 1.

(3) Is $d < 1$?

$$e^{rh} \left(1 - \sqrt{e^{\sigma^2 h} - 1}\right) < 1.$$

This expression will not always be true. If the term in parentheses is small, this condition can be violated. Thus, d is not always less than 1.

JKYABMD1

The mean and variance specifications for the physical process are

$$\begin{aligned}qu + (1 - q)d &= 1 + \alpha h \\ q(1 - q)(u/d)^2 &= \sigma^2 h.\end{aligned}$$

The mean can be written as

$$q(u - d) + d = 1 + \alpha h.$$

From the variance,

$$\begin{aligned}(u - d)^2 &= \frac{\sigma^2 h}{\sqrt{q(1 - q)}} \\ u - d &= \frac{\sigma \sqrt{h}}{q(1 - q)}.\end{aligned}$$

Substituting into the mean,

$$\begin{aligned}\frac{q\sigma\sqrt{h}}{\sqrt{q(1 - q)}} + d &= 1 + \alpha h \\ d &= 1 + \alpha h - \frac{q\sigma\sqrt{h}}{\sqrt{q(1 - q)}} \\ u &= d + \frac{\sigma\sqrt{h}}{\sqrt{q(1 - q)}} \\ u &= 1 + \alpha h - \frac{q\sigma\sqrt{h}}{\sqrt{q(1 - q)}} + \frac{\sigma\sqrt{h}}{\sqrt{q(1 - q)}} \\ u &= 1 + \alpha h + \frac{(1 - q)\sigma\sqrt{h}}{\sqrt{q(1 - q)}}\end{aligned}$$

Using $ud = 1$, we can get q :

$$ud = \left(1 + \alpha h + \frac{(1-q)\sigma\sqrt{h}}{\sqrt{q(1-q)}}\right) \left(1 + \alpha h + \frac{-q\sigma\sqrt{h}}{\sqrt{q(1-q)}}\right) = 1$$

$$(1 + \alpha h)^2 - (1 + \alpha h) \frac{q\sigma\sqrt{h}}{\sqrt{q(1-q)}} + (1 + \alpha h) \frac{(1-q)\sigma\sqrt{h}}{\sqrt{q(1-q)}} - \sigma^2 h = 1$$

$$(1 + \alpha h)^2 + \frac{(1 + \alpha h)\sigma\sqrt{h}(1-2q)}{\sqrt{q(1-q)}} = 1 + \sigma^2 h$$

$$\frac{(1 + \alpha h)\sigma\sqrt{h}(1-2q)}{\sqrt{q(1-q)}} = (1 + \sigma^2 h) - (1 + \alpha h)^2$$

$$(1 + \alpha h)\sigma\sqrt{h}(1-2q) = \sqrt{q(1-q)} \left((1 + \sigma^2 h) - (1 + \alpha h)^2 \right)$$

$$(1 + \alpha h)^2 \sigma^2 h (1-2q)^2 = q(1-q) \left((1 + \sigma^2 h) - (1 + \alpha h)^2 \right)^2$$

$$(1-2q)^2 = q(1-q) \frac{\left((1 + \sigma^2 h) - (1 + \alpha h)^2 \right)^2}{(1 + \alpha h)^2 \sigma^2 h}$$

$$1 - 4q + 4q^2 = (q - q^2) \left[\frac{(1 + \sigma^2 h) - (1 + \alpha h)^2}{(1 + \alpha h)\sigma\sqrt{h}} \right]^2.$$

Call the bracketed term m . Then

$$1 - q + 4q^2 = (q - q^2)m^2$$

$$1 - q + 4q^2 - qm^2 + q^2m^2 = 0$$

$$q^2(4 + m^2) + q(-4 - m^2) + 1 = 0.$$

Let $a = 4 + m^2$, giving

$$q^2 a + 4(-a) + 1 = 0.$$

Using the quadratic formula, the solution is

$$\begin{aligned} q &= \frac{1}{2} \pm \sqrt{\frac{a-4}{4a}} = \frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{a-4}{a}} \\ &= \frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{4+m^2-4}{4+m^2}} = \frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{m^2}{4+m^2}}. \end{aligned}$$

As it turns out, both solutions are correct and reasonable. JKY report the formula based on the minus solution. Thus, they report that

$$q = \frac{1}{2} - \frac{1}{2} \frac{m}{\sqrt{4+m^2}}, \quad m = \frac{(1 + \sigma^2 h) - (1 + \alpha h)^2}{(1 + \alpha h)\sigma\sqrt{h}}.$$

Risk neutralizing results in

$$u = 1 + rh + \frac{(1 - \pi)\sigma\sqrt{h}}{\sqrt{\pi(1 - \pi)}}, \quad d = 1 + rh + \frac{-\pi\sigma\sqrt{h}}{\sqrt{\pi(1 - \pi)}}$$

$$\pi^* = \frac{1}{2} - \frac{1}{2} \frac{m}{\sqrt{4+m^2}}, \quad m = \frac{(1 + \sigma^2 h) - (1 + rh)^2}{(1 + rh)\sigma\sqrt{h}}.$$

Now let us examine the limit of π^* .

$$\lim_{h \rightarrow 0} \pi^* = \lim_{h \rightarrow 0} \left(\frac{1}{2} \left(1 - \frac{m}{\sqrt{4+m^2}} \right) \right) = \frac{1}{2} - \frac{1}{2} \lim_{h \rightarrow 0} \left(\frac{m}{\sqrt{4+m^2}} \right).$$

Let us examine the limit of m :

$$\lim_{h \rightarrow 0} m = \lim_{h \rightarrow 0} \left(\frac{1 + \sigma^2 h - (1 + rh)^2}{(1 + rh)\sigma\sqrt{h}} \right) = \frac{0}{0}.$$

Using L'Hôpital's rule:

$$\begin{aligned} m &= \frac{f(h)}{g(h)} = \frac{(1 + \sigma^2 h) - (1 + rh)^2}{(1 + rh)\sigma\sqrt{h}} \\ f'(h) &= \sigma^2 - 2r(1 + rh) \\ g'(h) &= (1 + rh) \frac{\sigma}{2\sqrt{h}} + \sigma r\sqrt{h} \\ \frac{f'(h)}{g'(h)} &= \frac{\sigma^2 - 2r(1 + rh)}{(1 + rh) \frac{\sigma}{2\sqrt{h}} + \sigma r\sqrt{h}} = \frac{\sigma^2 - 2r(1 + rh)}{\frac{(1 + rh)\sigma + 2\sigma rh}{2\sqrt{h}}} = \frac{2\sqrt{h}(\sigma^2 - 2r(1 + rh))}{(1 + rh)\sigma + 2\sigma rh} \\ \lim_{h \rightarrow 0} \left(\frac{f'(h)}{g'(h)} \right) &= \frac{0}{\sigma} = 0. \end{aligned}$$

Thus,

$$\lim_{h \rightarrow 0} \pi^* = \frac{1}{2} - \frac{1}{2} \lim_{h \rightarrow 0} \left(\frac{m}{\sqrt{4+m^2}} \right) = \frac{1}{2} - \frac{1}{2} \left(\frac{0}{2} \right) = \frac{1}{2}.$$

Thus, the JKYABMD1 formula for π^* converges to $\frac{1}{2}$ in the limit. This is not the arbitrage-free formula for π . We must show that it converges to $\frac{1}{2}$ in the limit to verify that JKYABMD1's formula for π^* converges to the correct, arbitrage-free value. The formulas for u and d are repeated:

$$u = 1 + rh + \frac{(1 - \pi')\sigma\sqrt{h}}{\sqrt{\pi'(1 - \pi')}}, \quad d = 1 + rh + \frac{-\pi'\sigma\sqrt{h}}{\sqrt{\pi'(1 - \pi')}}.$$

We just proved that in the limit, π^* goes to $\frac{1}{2}$. So substitute $\frac{1}{2}$:

$$u = 1 + rh + \sigma\sqrt{h}, \quad d = 1 + rh + \sigma\sqrt{h}.$$

Now look at the arbitrage-free formula for π :

$$\pi = \frac{e^{rh} - d}{u - d} = \frac{e^{rh} - (1 + rh - \sigma\sqrt{h})}{(1 + rh + \sigma\sqrt{h}) - (1 + rh - \sigma\sqrt{h})} = \frac{e^{rh} - (1 + rh - \sigma\sqrt{h})}{2\sigma\sqrt{h}}.$$

The limit is

$$\lim_{h \rightarrow 0} \left(\frac{e^{rh} - (1 + rh - \sigma\sqrt{h})}{2\sigma\sqrt{h}} \right) = \frac{1 - 1}{0}.$$

Thus, using L'Hôpital's rule:

$$\pi = \frac{e^{rh} - (1 + rh - \sigma\sqrt{h})}{2\sigma\sqrt{h}} = \frac{f(h)}{g(h)}$$

$$f'(h) = re^{rh} - r + \frac{\sigma}{2\sqrt{h}}$$

$$g'(h) = \frac{2\sigma}{2\sqrt{h}}$$

$$\frac{f'(h)}{g'(h)} = \frac{re^{rh} - r + \frac{\sigma}{2\sqrt{h}}}{\frac{2\sigma}{2\sqrt{h}}} = \frac{2\sqrt{h}(re^{rh} - r) + \sigma}{2\sigma} = \frac{2\sqrt{h}(re^{rh} - r) + \sigma}{2\sigma}$$

$$\lim_{h \rightarrow 0} \left(\frac{2\sqrt{h}(re^{rh} - r) + \sigma}{2\sigma} \right) = \frac{\sigma}{2\sigma} = \frac{1}{2}.$$

Thus, the correct arbitrage-free risk neutral probability converges to $\frac{1}{2}$, and the JKYABMD1 risk neutral probability converges to $\frac{1}{2}$, so the JKYABMD1 model is arbitrage-free in the limit.

Anomalies:

(1a) Is $\pi^* < 1$?

The formula for π^* guarantees that it will be less than $\frac{1}{2}$. Thus, π is always less than 1.

(1b) Is $\pi^* > 0$?

Given the formula for π^* and m , we have

$$\frac{1}{2} - \frac{1}{2} \frac{m}{\sqrt{4+m^2}} > 0$$

$$\frac{1}{2} < \frac{1}{2} \frac{m}{\sqrt{4+m^2}}$$

$$1 > \frac{m}{\sqrt{4+m^2}}$$

$$\sqrt{4+m^2} < m$$

$$4+m^2 > m.$$

This π^* is always greater than 0.

(2) $u > 1$?

It is apparent from the formula for u that it always exceeds 1.

(3) $d < 1$?

From the formula for d , it is possible that d could exceed 1 if σ is small enough.

JKYRB2

The mean and variance specifications for the physical process are

$$\begin{aligned} q \ln u + (1-q) \ln d &= \mu h \\ q(1-q) [\ln(u/d)]^2 &= \sigma^2 h. \end{aligned}$$

The up and down parameters are the same as the in the generalized RBJRT models:

$$u = e^{\frac{\mu h + \frac{1-q}{\sqrt{q(1-q)}} \sigma \sqrt{h}}{\sqrt{q(1-q)}}}, \quad d = e^{\frac{\mu h - \frac{q}{\sqrt{q(1-q)}} \sigma \sqrt{h}}{\sqrt{q(1-q)}}}.$$

JKY impose the specification

$$ud = e^{2\alpha h}.$$

Thus,

$$\begin{aligned} ud &= e^{\frac{2\mu h + \frac{\sigma \sqrt{h}(1-2q)}{\sqrt{q(1-q)}}}{\sqrt{q(1-q)}}} = e^{2\alpha h} \\ 2\mu h + \frac{\sigma \sqrt{h}(1-2q)}{\sqrt{q(1-q)}} &= 2\alpha h. \end{aligned}$$

Now we substitute $\alpha - \sigma^2/2$ for μ so that $2\mu h = 2(\alpha - \sigma^2/2)h$. Then

$$\begin{aligned} 2(\alpha - \sigma^2/2)h + \frac{\sigma \sqrt{h}(1-2q)}{\sqrt{q(1-q)}} &= 2\alpha h \\ -\sigma^2 h + \frac{\sigma \sqrt{h}(1-2q)}{\sqrt{q(1-q)}} &= 0 \\ \frac{\sigma \sqrt{h}(1-2q)}{\sqrt{q(1-q)}} &= \sigma^2 h. \end{aligned}$$

Note here that because the RHS must be positive, we require that $q < 1/2$. This will identify which root to use when we solve a quadratic equation.

Continuing with the above expression:

$$\begin{aligned} \frac{\sigma^2 h(1-2q)^2}{q(1-q)} &= \sigma^4 h^2 \\ \frac{(1-2q)^2}{q(1-q)} &= \sigma^2 h \\ (1-2q)^2 &= \sigma^2 hq(1-q) \\ 1 - 4q + 4q^2 &= \sigma^2 hq - \sigma^2 hq^2 \\ 4q^2 + \sigma^2 hq^2 - 4q - \sigma^2 hq + 1 &= 0 \\ q^2(4 + \sigma^2 h) + q(-4 - \sigma^2 h) + 1 &= 0. \end{aligned}$$

This is a quadratic equation of the form $aq^2 + bq + c = 0$ where $a = 4 + \sigma^2 h$, $b = -4 - \sigma^2 h$, and $c = 1$. The solution is the same as the derivation for JKYABMD1 except that we know that q must be less than $1/2$, so we must use the negative root:

$$q = \frac{1}{2} - \frac{1}{2} \frac{m}{\sqrt{4+m^2}}, \quad m = \sigma\sqrt{h}.$$

The risk neutral version requires that we replace μ with $r - \sigma^2/2$.

$$u = e^{(r-\sigma^2/2)h + \frac{1-\pi'}{\sqrt{\pi'(1-\pi')}}\sigma\sqrt{h}}, \quad d = e^{(r-\sigma^2/2)h - \frac{\pi'}{\sqrt{\pi'(1-\pi')}}\sigma\sqrt{h}}$$

$$\pi^* = \frac{1}{2} - \frac{1}{2} \frac{m}{\sqrt{4+m^2}}, \quad m = \sigma\sqrt{h}.$$

To find the limit of π^* , we will need the limit of m . From the formula for m , we see that the limit of m is 0. Thus, the limit of π^* is obviously $1/2$.

As noted, the JKYRB2 model does not specify the correct arbitrage-free mean formula. Hence, its formula for π^* admits arbitrage. We need to examine the limit of the correct arbitrage-free formula for π .

$$\pi = \frac{e^{rh} - d}{u - d} = \frac{e^{rh} - \left(e^{(r-\sigma^2/2)h - \frac{\pi'}{\sqrt{\pi'(1-\pi')}}\sigma\sqrt{h}} \right)}{e^{(r-\sigma^2/2)h + \frac{1-\pi'}{\sqrt{\pi'(1-\pi')}}\sigma\sqrt{h}} - \left(e^{(r-\sigma^2/2)h - \frac{\pi'}{\sqrt{\pi'(1-\pi')}}\sigma\sqrt{h}} \right)}.$$

We showed that π^* converges to $1/2$ in the limit. Thus, we can essentially write this formula as

$$\pi = \frac{e^{rh} - \left(e^{(r-\sigma^2/2)h - \sigma\sqrt{h}} \right)}{e^{(r-\sigma^2/2)h + \sigma\sqrt{h}} - \left(e^{(r-\sigma^2/2)h - \sigma\sqrt{h}} \right)}.$$

The limit is

$$\lim_{h \rightarrow 0} \pi = \lim_{h \rightarrow 0} \left(\frac{e^{rh} - \left(e^{(r-\sigma^2/2)h - \sigma\sqrt{h}} \right)}{e^{(r-\sigma^2/2)h + \sigma\sqrt{h}} - \left(e^{(r-\sigma^2/2)h - \sigma\sqrt{h}} \right)} \right) = \frac{1-1}{1-1}.$$

Thus, using L'Hôpital's rule:

$$\begin{aligned}
\pi &= \frac{e^{rh} - e^{(r-\sigma^2/2)h - \sigma\sqrt{h}}}{e^{(r-\sigma^2/2)h + \sigma\sqrt{h}} - e^{(r-\sigma^2/2)h - \sigma\sqrt{h}}} = \frac{f(h)}{g(h)} \\
f'(h) &= re^{rh} - e^{(r-\sigma^2/2)h - \sigma\sqrt{h}} \left(r - \sigma^2/2 - \frac{\sigma}{2\sqrt{h}} \right) \\
&= \frac{2\sqrt{h} \left(re^{rh} - e^{(r-\sigma^2/2)h - \sigma\sqrt{h}} (r - \sigma^2/2) \right) + \sigma e^{(r-\sigma^2/2)h - \sigma\sqrt{h}}}{2\sqrt{h}} \\
g'(h) &= e^{(r-\sigma^2/2)h + \sigma\sqrt{h}} \left(r - \sigma^2/2 + \frac{\sigma}{2\sqrt{h}} \right) - e^{(r-\sigma^2/2)h - \sigma\sqrt{h}} \left(r - \sigma^2/2 - \frac{\sigma}{2\sqrt{h}} \right) \\
&= \frac{2\sqrt{h} \left(e^{(r-\sigma^2/2)h + \sigma\sqrt{h}} (r - \sigma^2/2) + \sigma e^{(r-\sigma^2/2)h + \sigma\sqrt{h}} - 2\sqrt{h} e^{(r-\sigma^2/2)h - \sigma\sqrt{h}} (r - \sigma^2/2) + \sigma e^{(r-\sigma^2/2)h - \sigma\sqrt{h}} \right)}{2\sqrt{h}} \\
&\quad \frac{2\sqrt{h} \left(re^{rh} - e^{(r-\sigma^2/2)h - \sigma\sqrt{h}} (r - \sigma^2/2) \right) + \sigma e^{(r-\sigma^2/2)h - \sigma\sqrt{h}}}{2\sqrt{h}} \\
\frac{f'(h)}{g'(h)} &= \frac{2\sqrt{h}}{2\sqrt{h} \left(e^{(r-\sigma^2/2)h + \sigma\sqrt{h}} (r - \sigma^2/2) + \sigma e^{(r-\sigma^2/2)h + \sigma\sqrt{h}} - 2\sqrt{h} e^{(r-\sigma^2/2)h - \sigma\sqrt{h}} (r - \sigma^2/2) + \sigma e^{(r-\sigma^2/2)h - \sigma\sqrt{h}} \right)} \\
&= \frac{2\sqrt{h} \left(re^{rh} - e^{(r-\sigma^2/2)h - \sigma\sqrt{h}} (r - \sigma^2/2) \right) + \sigma e^{(r-\sigma^2/2)h - \sigma\sqrt{h}}}{2\sqrt{h} \left(e^{(r-\sigma^2/2)h + \sigma\sqrt{h}} (r - \sigma^2/2) + \sigma e^{(r-\sigma^2/2)h + \sigma\sqrt{h}} - 2\sqrt{h} e^{(r-\sigma^2/2)h - \sigma\sqrt{h}} (r - \sigma^2/2) + \sigma e^{(r-\sigma^2/2)h - \sigma\sqrt{h}} \right)} \\
&= \lim_{h \rightarrow 0} \left(\frac{2\sqrt{h} \left(re^{rh} - e^{(r-\sigma^2/2)h - \sigma\sqrt{h}} (r - \sigma^2/2) \right) + \sigma e^{(r-\sigma^2/2)h - \sigma\sqrt{h}}}{2\sqrt{h} \left(e^{(r-\sigma^2/2)h + \sigma\sqrt{h}} (r - \sigma^2/2) + \sigma e^{(r-\sigma^2/2)h + \sigma\sqrt{h}} - 2\sqrt{h} e^{(r-\sigma^2/2)h - \sigma\sqrt{h}} (r - \sigma^2/2) + \sigma e^{(r-\sigma^2/2)h - \sigma\sqrt{h}} \right)} \right) \\
&= \frac{0 + \sigma}{0 + \sigma - 0 + \sigma} = \frac{\sigma}{2\sigma} = \frac{1}{2}.
\end{aligned}$$

Thus, the arbitrage-free value of π converges to $1/2$. Because the JKYRB2 formula for π also converges to $1/2$, the JKYRB2 model is arbitrage-free in the limit.

Anomalies:

(1a) Is $\pi^* < 1$?

It is apparent from the formula for π^* that it is less than $1/2$, so π is always less than 1.

(1b) Is $\pi^* > 0$?

The proof for JKYABMD1 is applicable to this case as well. Thus, π^* is always less than 0.

(2) Is $u > 1$?

$$e^{rh} \left(1 + \frac{1 - \pi'}{\sqrt{\pi'(1 - \pi')}} \sqrt{e^{\sigma^2 h} - 1} \right) > 1.$$

The expression in parentheses is greater than 1, so u is always greater than 1.

(3) Is $d < 1$?

$$e^{rh} \left(1 - \frac{\pi}{\sqrt{\pi(1-\pi)}} \sqrt{e^{\sigma^2 h} - 1} \right) > 1.$$

This statement is not necessarily true because a small enough volatility can make the subtracted term be sufficiently small to keep the LHS above 1. Thus, d can be greater than 1.

JKYABMC2

The mean and variance specifications for the physical process are

$$\begin{aligned}qu + (1 - q)d &= e^{\alpha h} \\ q(1 - q)(u - d)^2 &= e^{2\alpha h}(e^{\sigma^2 h} - 1).\end{aligned}$$

Write these as:

$$\begin{aligned}q(u - d) + d &= e^{\alpha h} \\ (u - d)^2 &= \frac{e^{2\alpha h}(e^{\sigma^2 h} - 1)}{q(1 - q)} \\ u - d &= \sqrt{\frac{e^{2\alpha h}(e^{\sigma^2 h} - 1)}{q(1 - q)}} = \frac{e^{\alpha h}}{\sqrt{q(1 - q)}}\sqrt{(e^{\sigma^2 h} - 1)}.\end{aligned}$$

Working with the mean

$$\begin{aligned}q \frac{e^{\alpha h}}{\sqrt{q(1 - q)}}\sqrt{e^{\sigma^2 h} - 1} + d &= e^{\alpha h} \\ d &= e^{\alpha h} - \frac{e^{\alpha h} q}{\sqrt{q(1 - q)}}\sqrt{e^{\sigma^2 h} - 1} \\ d &= e^{\alpha h} \left(1 - \frac{q}{\sqrt{q(1 - q)}}\sqrt{e^{\sigma^2 h} - 1} \right) \\ u &= d + \frac{e^{\alpha h}}{\sqrt{q(1 - q)}}\sqrt{e^{\sigma^2 h} - 1} \\ &= e^{\alpha h} \left(1 - \frac{\sqrt{e^{\sigma^2 h} - 1}}{\sqrt{q(1 - q)}} \right) - \frac{qe^{\alpha h}}{\sqrt{q(1 - q)}}\sqrt{e^{\sigma^2 h} - 1} \\ u &= e^{\alpha h} \left(1 + \frac{(1 - q)}{\sqrt{q(1 - q)}}\sqrt{e^{\sigma^2 h} - 1} \right).\end{aligned}$$

Now we need to solve for q . They impose the specification

$$ud = e^{2\alpha h}.$$

Thus,

$$ud = e^{\alpha h} \left(1 + \frac{1 - q}{q\sqrt{(1 - q)}}\sqrt{e^{\sigma^2 h} - 1} \right) e^{\alpha h} \left(1 - \frac{q}{\sqrt{(1 - q)}}\sqrt{e^{\sigma^2 h} - 1} \right) = e^{2\alpha h}.$$

Therefore,

$$\begin{aligned}
& e^{2\alpha h} \left(1 + \frac{1-q}{\sqrt{q(1-q)}} \sqrt{e^{\sigma^2 h} - 1} \right) \left(1 - \frac{q}{\sqrt{q(1-q)}} \sqrt{e^{\sigma^2 h} - 1} \right) = e^{2\alpha h} \\
& e^{2\alpha h} \left(1 - \frac{q}{\sqrt{q(1-q)}} \sqrt{e^{\sigma^2 h} - 1} + \frac{1-q}{\sqrt{q(1-q)}} \sqrt{e^{\sigma^2 h} - 1} - \frac{q(1-q)}{q(1-q)} (e^{\sigma^2 h} - 1) \right) = 1 \\
& \left(1 + \frac{\sqrt{e^{\sigma^2 h} - 1}}{\sqrt{q(1-q)}} (1-2q) - (e^{\sigma^2 h} - 1) \right) = 1.
\end{aligned}$$

Because of the constraint on ud , the term in brackets must equal 1. Therefore,

$$\frac{\sqrt{e^{\sigma^2 h} - 1}(1-2q)}{\sqrt{q(1-q)}} = e^{\sigma^2 h} - 1.$$

Note from the above that if $q > \frac{1}{2}$ the LHS will be negative, but the RHS is clearly positive. Thus, we must have $q < \frac{1}{2}$. This will tell us to use the negative root when we solve the quadratic equation. Continuing:

$$\begin{aligned}
& \frac{\sqrt{e^{\sigma^2 h} - 1}(1-2q)}{\sqrt{q(1-q)}} = e^{\sigma^2 h} - 1 \\
& \sqrt{e^{\sigma^2 h} - 1}(1-2q) = (e^{\sigma^2 h} - 1)\sqrt{q(1-q)} \\
& (e^{\sigma^2 h} - 1)(1-2q)^2 = (e^{\sigma^2 h} - 1)^2 q(1-q) \\
& (1-2q)^2 = q(1-q)(e^{\sigma^2 h} - 1) \\
& 1 - 4q + 4q^2 = (q - q^2)(e^{\sigma^2 h} - 1) \\
& 1 - 4q + 4q^2 = q(e^{\sigma^2 h} - 1) - q^2(e^{\sigma^2 h} - 1) \\
& 4q^2 + (e^{\sigma^2 h} - 1)q^2 - 4q - q(e^{\sigma^2 h} - 1) + 1 = 0 \\
& q^2(4 + (e^{\sigma^2 h} - 1)) + q(-4(e^{\sigma^2 h} - 1)) + 1 = 0.
\end{aligned}$$

This is a quadratic equation just like the ones associated with the JKYABMD1 and JKYRB2 models. The solution is

$$q = \frac{1}{2} - \frac{1}{2} \frac{m}{\sqrt{4+m^2}}, \quad m = \sqrt{e^{\sigma^2 h} - 1}.$$

To risk neutralize, change α to r .

$$\begin{aligned}
& u = e^{rh} \left(1 + \frac{(1-q)}{\sqrt{q(1-q)}} \sqrt{e^{\sigma^2 h} - 1} \right), \quad d = e^{rh} \left(1 - \frac{q}{\sqrt{q(1-q)}} \sqrt{e^{\sigma^2 h} - 1} \right) \\
& \pi^* = \frac{1}{2} - \frac{1}{2} \frac{m}{\sqrt{4+m^2}}, \quad m = \sqrt{e^{\sigma^2 h} - 1}.
\end{aligned}$$

To find the limit of π^* , we will need the limit of m . From the formula for m , we see that the limit of m is 0. Thus, the limit of π^* is obviously $\frac{1}{2}$. We also need to determine if the arbitrage-free formula for π converges to $\frac{1}{2}$ in the limit.

$$\pi = \frac{e^{rh} - d}{u - d} = \frac{e^{rh} - \left(e^{rh} \left(1 - \frac{\pi'}{\sqrt{\pi'(1-\pi')}} \sqrt{e^{\sigma^2 h} - 1} \right) \right)}{e^{rh} \left(1 + \frac{(1-\pi')}{\sqrt{\pi'(1-\pi')}} \sqrt{e^{\sigma^2 h} - 1} \right) - e^{rh} \left(1 - \frac{\pi'}{\sqrt{\pi'(1-\pi')}} \sqrt{e^{\sigma^2 h} - 1} \right)}.$$

We know that π^* converges to $\frac{1}{2}$ so we can effectively substitute $\frac{1}{2}$, giving

$$\lim_{h \rightarrow 0} \pi = e^{rh} \frac{e^{rh} - \left(e^{rh} \left(1 - \sqrt{e^{\sigma^2 h} - 1} \right) \right)}{e^{rh} \left(1 + \sqrt{e^{\sigma^2 h} - 1} \right) - e^{rh} \left(1 - \sqrt{e^{\sigma^2 h} - 1} \right)} = \frac{e^{rh} \sqrt{e^{\sigma^2 h} - 1}}{2e^{rh} \sqrt{e^{\sigma^2 h} - 1}} = \frac{1}{2}.$$

Therefore, we see that the correct arbitrage-free risk neutral probability is $\frac{1}{2}$ in the limit with these chosen values of u and d . Because π^* converges to $\frac{1}{2}$ in the limit and π also converges to $\frac{1}{2}$ in the limit, the JKYABMC2 model is arbitrage-free in the limit.

Anomalies:

(1a) Is $\pi^* < 1$?

We determined that π^* must be less than $\frac{1}{2}$, so π^* is always less than 1.

(1b) Is $\pi^* > 0$?

The proof for JKYABMD1 is applicable to this case as well. Thus, π^* is always greater than 0.

(2) Is $u > 1$?

$$e^{rh} \left(1 + \frac{(1-\pi')}{\sqrt{\pi'(1-\pi')}} \sqrt{e^{\sigma^2 h} - 1} \right) > 1.$$

The expression in parentheses is greater than 1, so u is always greater than 1.

(3) Is $d < 1$?

$$e^{rh} \left(1 - \frac{\pi'}{\sqrt{\pi'(1-\pi')}} \sqrt{e^{\sigma^2 h} - 1} \right) < 1.$$

This statement is not necessarily true because a small enough volatility can make the term subtracted be sufficiently small to keep the LHS above 1. Thus, d is not always less than 1.

JKYABMD2c

The mean and variance specifications for the physical process are

$$\begin{aligned}qu + (1 - q)d &= 1 + \alpha h \\ q(1 - q)(u - d)^2 &= \sigma^2 h.\end{aligned}$$

The mean can be written as

$$q(u - d) + d = 1 + \alpha h.$$

Then

$$\begin{aligned}(u - d)^2 &= \frac{\sigma^2 h}{\sqrt{q(1 - q)}} \\ u - d &= \frac{\sigma \sqrt{h}}{\sqrt{q(1 - q)}}.\end{aligned}$$

Using the mean,

$$\begin{aligned}\frac{q\sigma\sqrt{h}}{\sqrt{q(1 - q)}} + d &= 1 + \alpha h \\ d &= 1 + \alpha h - \frac{q\sigma\sqrt{h}}{\sqrt{q(1 - q)}} \\ u &= d + \frac{\sigma\sqrt{h}}{\sqrt{q(1 - q)}} \\ u &= 1 + \alpha h - \frac{q\sigma\sqrt{h}}{\sqrt{q(1 - q)}} + \frac{\sigma\sqrt{h}}{\sqrt{q(1 - q)}} \\ u &= 1 + \alpha h + \frac{(1 - q)\sigma\sqrt{h}}{\sqrt{q(1 - q)}}.\end{aligned}$$

To obtain a value for q , they assume $ud = e^{2\alpha h}$. JKY give the solution as

JKY solution:

$$q = \frac{1}{2} - \frac{1}{2} \frac{m}{\sqrt{4 + m^2}}, \quad m = \frac{\sigma\sqrt{h}}{1 + \alpha h}.$$

The formula for q is correct, but the formula for m requires another assumption. The precise derivation follows. We have

$$ud = \left(1 + \alpha h + \frac{(1 - q)\sigma\sqrt{h}}{\sqrt{q(1 - q)}}\right) \left(1 + \alpha h + \frac{-q\sigma\sqrt{h}}{\sqrt{q(1 - q)}}\right) = e^{2\alpha h}.$$

Therefore,

$$ud = (1 + \alpha h)^2 + \frac{(1 + \alpha h)\sigma\sqrt{h}(1 - 2q)}{\sqrt{q(1 - q)}} - \sigma^2 h = e^{2\alpha h}$$

$$\begin{aligned} \frac{(1 + \alpha h)\sigma\sqrt{h}(1 - 2q)}{\sqrt{q(1 - q)}} &= e^{2\alpha h} + \sigma^2 h - (1 + \alpha h)^2 \\ (1 + \alpha h)\sigma\sqrt{h}(1 - 2q) &= \sqrt{q(1 - q)}(e^{2\alpha h} + \sigma^2 h - (1 + \alpha h)^2) \\ (1 + \alpha h)^2 \sigma^2 h (1 - 2q)^2 &= q(1 - q)W^2, \end{aligned}$$

where

$$W = e^{2\alpha h} + \sigma^2 h - (1 + \alpha h)^2.$$

Proceeding,

$$(1 - 2q)^2 = q(1 - q) \frac{W^2}{(1 + \alpha h)^2 \sigma^2 h}.$$

Let

$$m = \frac{\sigma^2 h + e^{2\alpha h} - (1 + \alpha h)^2}{(1 + \alpha h)\sigma\sqrt{h}}.$$

The JKY solution for m is obtained by using the approximation $e^x \cong 1 + x$. The precise solution contains an addition term in the numerator, $e^{2\alpha h} - (1 + \alpha h)^2$, that corrects for the approximation. The precise solution is obtained as:

$$\begin{aligned} 1 - 4q + 4q^2 &= (q - q^2)m^2 \\ 1 - 4q + 4q^2 - qm^2 + q^2m^2 &= 0 \\ q^2(4 + m^2) + q(-4 - m^2) + 1 &= 0. \end{aligned}$$

This is a quadratic equation as in the other JKY models. The solution is

$$q = \frac{1}{2} - \frac{1}{2} \frac{m}{\sqrt{4 + m^2}}.$$

Of course, there are two roots to the quadratic equation, but we can deduce that only the negative root applies. Recall that above, we had the equation

$$\begin{aligned} (1 + \alpha h)^2 \sigma^2 h (1 - 2q)^2 &= q(1 - q)W^2 \text{ where} \\ W &= e^{2\alpha h} + \sigma^2 h - (1 + \alpha h)^2. \end{aligned}$$

Using a Taylor-series expansion, we can show that W is positive:

$$\begin{aligned} e^{2\alpha h} - (1 + \alpha h)^2 &= 1 + 2\alpha h + \frac{4\alpha^2 h^2}{2} + \dots - (1 + 2\alpha h + \alpha^2 h^2) \\ &= \alpha^2 h^2 + \dots \end{aligned}$$

Taking the square root from an earlier expression:

$$(1 + \alpha h)\sigma\sqrt{h}(1 - 2q) = \sqrt{q(1 - q)}W.$$

We know that the RHS is positive. The LHS can be positive only if $1 > 2q$, which means that $q < \frac{1}{2}$. Therefore, we require the negative root to make q be less than $\frac{1}{2}$.

Risk neutralizing gives solutions of

$$u = 1 + rh + \frac{(1 - \pi)\sigma\sqrt{h}}{\sqrt{\pi(1 - \pi)}}, \quad d = 1 + rh + \frac{-\pi\sigma\sqrt{h}}{\sqrt{\pi(1 - \pi)}}$$

$$\pi^* = \frac{1}{2} - \frac{1}{2} \frac{m}{\sqrt{4 + m^2}}, \quad m = \frac{e^{2rh} - (1 + rh)^2 + \sigma^2 h}{(1 + rh)\sigma\sqrt{h}}.$$

Now consider the limit of π^* . We will need to know the limit of m .

$$\lim_{h \rightarrow 0} m = \lim_{h \rightarrow 0} \left(\frac{e^{2rh} - (1 + rh)^2 + \sigma^2 h}{(1 + rh)\sigma\sqrt{h}} \right) = \frac{1 - 1}{0}.$$

Thus, we use L'Hôpital's rule:

$$\lim_{h \rightarrow 0} m = \lim_{h \rightarrow 0} \left(\frac{f(h)}{g(h)} \right)$$

$$f(h) = \sigma^2 h + e^{2rh} - (1 + rh)^2$$

$$f'(h) = \sigma^2 + 2re^{2rh} - 2(1 + rh)r$$

$$g(h) = (1 + rh)\sigma\sqrt{h}$$

$$g'(h) = \frac{(1 + rh)\sigma}{2\sqrt{h}} + r\sigma\sqrt{h}$$

$$\frac{f'(h)}{g'(h)} = \frac{\sigma^2 + 2re^{2rh} - 2(1 + rh)r}{\frac{(1 + rh)\sigma}{2\sqrt{h}} + r\sigma\sqrt{h}} \quad \text{He}$$

$$= \frac{\sigma^2 + 2re^{2rh} - 2(1 + rh)r}{\frac{(1 + rh)\sigma + \sigma\sqrt{hr}\sqrt{h}}{2\sqrt{h}}} = \frac{2\sqrt{h}(e^{2rh} 2r + \sigma^2 - 2(1 + rh)r)}{(1 + rh)\sigma + \sigma\sqrt{hr}\sqrt{h}}$$

$$\lim_{h \rightarrow 0} \left(\frac{2\sqrt{h}(e^{2rh} 2r + \sigma^2 - 2(1 + rh)r)}{(1 + rh)\sigma + \sigma hr} \right) = \frac{0}{\sigma} = 0.$$

nce, the limit of π^* is $\frac{1}{2}$. We need to examine the limit of the correct, arbitrage-free value of π^* :

$$\pi = \frac{e^{rh} - \left(1 + rh + \frac{-\pi'\sigma\sqrt{h}}{\sqrt{\pi'(1 - \pi')}} \right)}{\left(1 + rh + \frac{(1 - \pi')\sigma\sqrt{h}}{\sqrt{\pi'(1 - \pi')}} \right) - \left(1 + rh + \frac{-\pi\sigma\sqrt{h}}{\sqrt{\pi'(1 - \pi')}} \right)}.$$

We know that π^* converges to $\frac{1}{2}$ in the limit. Therefore, we can effectively substitute and get:

$$\pi = \frac{e^{rh} - (1 + rh - \sigma\sqrt{h})}{(1 + rh + \sigma\sqrt{h}) - (1 + rh - \sigma\sqrt{h})} = \frac{e^{rh} - (1 + rh - \sigma\sqrt{h})}{2\sigma\sqrt{h}}.$$

The limit is

$$\lim_{h \rightarrow 0} \pi = \lim_{h \rightarrow 0} \left(\frac{e^{rh} - (1 + rh - \sigma\sqrt{h})}{2\sigma\sqrt{h}} \right) = \frac{1-1}{0}.$$

Thus, using L'Hôpital's rule:

$$\begin{aligned} \pi &= \frac{e^{rh} - (1 + rh - \sigma\sqrt{h})}{2\sigma\sqrt{h}} = \frac{f(h)}{g(h)} \\ f'(h) &= re^{rh} - r + \frac{\sigma}{2\sqrt{h}} = \frac{\sqrt{h}(re^{rh} - r) + (1/2)\sigma}{\sqrt{h}} \\ g'(h) &= \frac{\sigma}{\sqrt{h}} \\ \frac{f'(h)}{g'(h)} &= \frac{\frac{\sqrt{h}(re^{rh} - r) + (1/2)\sigma}{\sqrt{h}}}{\frac{\sigma}{\sqrt{h}}} = \frac{\sqrt{h}(re^{rh} - r) + (1/2)\sigma}{\sigma} \\ \lim_{h \rightarrow 0} &= \lim_{h \rightarrow 0} \left(\frac{\sqrt{h}(re^{rh} - r) + (1/2)\sigma}{\sigma} \right) = \frac{1}{2}. \end{aligned}$$

Thus, the arbitrage-free value of π is $\frac{1}{2}$. Because π^* converges to $\frac{1}{2}$, the JKYABMD2c model is arbitrage-free in the limit.

Anomalies

(1a) Is $\pi^* < 1$?

We have already shown that $\pi^* < \frac{1}{2}$, so π^* is always less than 1.

(1b) Is $\pi^* > 0$?

The proof for JKYABMD1 is applicable to this case as well. Thus, π^* is always greater than 0.

(2) Is $u > 1$?

It is apparent from the formula that u is always greater than 1.

(3) Is $d < 1$?

It is apparent from the formula that a very small sigma can make d be greater than 1. Thus, d is not always less than 1.

JKYABMD3

The mean and variance specifications for the physical process are

$$\begin{aligned}qu + (1 - q)d &= 1 + \alpha h \\ q(1 - q)(u - d)^2 &= \sigma^2 h.\end{aligned}$$

The model assumes that $q = \frac{1}{2}$. Thus,

$$\begin{aligned}\frac{1}{2}u + \frac{1}{2}d &= 1 + \alpha h \\ \frac{1}{2}\left(\frac{1}{2}\right)(u - d)^2 &= \sigma^2 h \\ u + d &= 2(1 + \alpha h) \\ \frac{1}{4}(u - d)^2 &= \sigma^2 h \\ (u - d)^2 &= 4\sigma^2 h \\ u - d &= 2\sigma\sqrt{h}.\end{aligned}$$

So we have

$$\begin{aligned}u + d &= 2(1 + \alpha h) \\ u - d &= 2\sigma\sqrt{h} \\ d &= 2(1 + \alpha h) - u.\end{aligned}$$

Therefore,

$$\begin{aligned}u - d &= u - (2(1 + \alpha h) - u) = 2\sigma\sqrt{h} \\ 2u - 2(1 + \alpha h) &= 2\sigma\sqrt{h} \\ u - (1 + \alpha h) &= \sigma\sqrt{h} \\ u &= 1 + \alpha h + \sigma\sqrt{h}.\end{aligned}$$

To obtain d :

$$\begin{aligned}d &= 2(1 + \alpha h) - (1 + \alpha h + \sigma\sqrt{h}) = 2(1 + \alpha h) - (1 + \alpha h) - \sigma\sqrt{h} \\ &= 1 + \alpha h - \sigma\sqrt{h}.\end{aligned}$$

To risk neutralize the model, we have

$$\begin{aligned}u &= 1 + rh + \sigma\sqrt{h}, \quad d = 1 + rh - \sigma\sqrt{h} \\ \pi^* &= \frac{1}{2}.\end{aligned}$$

The risk neutral probability is arbitrarily set at $\frac{1}{2}$, but it does not prohibit arbitrage for finite time steps because the mean constraint is not the arbitrage-free constraint. Obviously, it is equal to $\frac{1}{2}$ in the limit. We need to show that the correct arbitrage-free value of π is $\frac{1}{2}$ in the limit.

$$\pi = \frac{e^{rh} - (1 + rh - \sigma\sqrt{h})}{(1 + rh + \sigma\sqrt{h}) - (1 + rh - \sigma\sqrt{h})} = \frac{e^{rh} - (1 + rh - \sigma\sqrt{h})}{2\sigma\sqrt{h}}.$$

Taking the limit:

$$\lim_{h \rightarrow 0} \pi = \lim_{h \rightarrow 0} \left(\frac{e^{rh} - (1 + rh - \sigma\sqrt{h})}{2\sigma\sqrt{h}} \right) = \frac{1-1}{0}.$$

Thus, using L'Hôpital's rule:

$$\begin{aligned} \pi &= \frac{e^{rh} - (1 + rh - \sigma\sqrt{h})}{2\sigma\sqrt{h}} = \frac{f(h)}{g(h)} \\ f'(h) &= re^{rh} - r + \frac{\sigma}{2\sqrt{h}} = \frac{\sqrt{h}(re^{rh} - r) + (1/2)\sigma}{\sqrt{h}} \\ g'(h) &= \frac{\sigma}{\sqrt{h}} \\ \frac{f'(h)}{g'(h)} &= \frac{\frac{\sqrt{h}(re^{rh} - r) + (1/2)\sigma}{\sqrt{h}}}{\frac{\sigma}{\sqrt{h}}} = \frac{\sqrt{h}(re^{rh} - r) + (1/2)\sigma}{\sigma} \\ \lim_{h \rightarrow 0} \left(\frac{f'(h)}{g'(h)} \right) &= \lim_{h \rightarrow 0} \left(\frac{\sqrt{h}(re^{rh} - r) + (1/2)\sigma}{\sigma} \right) = \frac{1}{2}. \end{aligned}$$

Because π converges to $1/2$ and $\pi^* = 1/2$, the model is arbitrage-free in the limit.

Anomalies:

(1a) Is $\pi^* < 1$?

Since $\pi^* = 1/2$, it is always less than 1.

(1b) Is $\pi^* > 0$?

Since $\pi^* = 1/2$, it is always greater than 0.

(2) Is $u > 1$?

It is obvious from the formula that u is always greater than 1.

(3) Is $d < 1$?

It is obvious from the formula that a small enough σ can result in $d > 1$. Thus, d is not always less than 1.