# TEACHING NOTE 18-03:

# PRICING INTEREST RATE SWAPS WITH BONDS VS. FORWARD RATES

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For swaps with no credit risk in which the appropriate discount rate is the rate to which the floating leg is pegged, pricing can be done by using either a pair of bonds or with forward rates. In this note, we demonstrate the equivalence of these two methods. Let us start by specifying a swap with payments at times $t = 1, 2, \ldots, T$. We shall assume the swap is a pay-fixed, receive-floating swap, which shall henceforth be simply referred to as a pay-fixed swap. Let the swap be based on an underlying rate, $L$, ostensibly representing LIBOR, though it could be any market rate with the same calculation convention.[1] At the issue date, the LIBOR term structure is specified as $L_{0,1}, L_{0,2}, \ldots, L_{0,T}$, which represents LIBORs observed at time 0 for the period to time $t$, $t = 1, 2, \ldots, T$. We assume this rate is LIBOR for a fixed number of days, say 90, but that each settlement period can have a different number of days. Denote the number of days divided by 360 in settlement period $t$ as $m_t$, and bear in mind that $t$ is the end of the settlement period, that is, the point at which the payment is made. Thus, we follow the standard LIBOR convention of advance set, settle in arrears. Thus, the rate for $t^{th}$ payment is set at time $t$-1. For $t = 1$, the first payment, the rate is set at $t - 1 = 0$. Therefore, the rate $L_{0,1}$, observed at time 0 sets the rate for the payment at $t = 1$. We shall also need to denote rates later during the life of the swap. For example, the one-period rate at time $t$ is $L_{t,t+1}$.

For discounting purposes, we shall need the number of days divided by 360 from time 0 to time $t$. This cumulative number of days divided by 360 will be denoted as $h_{0,t}$, which is formally defined as

$$h_{0,t} = \sum_{j=1}^{t} m_j \quad .$$

That is, the number of days divided by 360 from time 0 to time $t$ is the sum of the number of days in each settlement period to that point. At other times, we might wish to know the number of days divided by 360 from some arbitrary time $w$ to time $z$, which would be

$$h_{w,z} = \sum_{j=w+1}^{z} m_j$$

The discount factor for time 0 to time $t$ is, therefore,

[1]We mean that the interest calculation is add-on and based on the number of days divided by 360, which is the LIBOR convention, as we show later.

$$\frac{1}{1+L_{o,t}h_{0,t}}$$

And in general, the discount factor for time *w* to time *z* is

$$\frac{1}{1+L_{w,z}h_{w,z}}$$

Let *R* be the fixed rate on the swap. Thus, the $t^{th}$ payment from the perspective of the fixed payer will be

$$(L_{t-1,t}-R)m_t$$

We now proceed to demonstrate swap pricing.

**Forward Rate Pricing**

In pricing the swap via forward rates, we treat the unknown future floating payments as represented by the forward rates. This point is often stated as an assumption, but it is not an assumption, as we shall show. It is a condition that is required by the principle of no arbitrage. First, let us specify the unknown future floating swap payments as forward rates. The first payment, of course, is the spot rate at time 0, so that floating payment will be at $L_{0,1}$. The second floating payment will be at the rate $L_{1,2}$, but of course, this rate is unknown at time 0, and we use the forward rate in its place. Denote $F_{0,1,2}$ as the forward rate observed at time 0 for the period from time 1 to time 2. It is defined as

$$\begin{aligned} F_{0,1,2} &= \left(\left(\frac{1+L_{0,2}h_{0,2}}{1+L_{0,1}h_{0,1}}\right)-1\right)\left(\frac{1}{h_{0,2}-h_{0,1}}\right) \\ &= \left(\left(\frac{1+L_{0,2}h_{0,2}}{1+L_{0,1}h_{0,1}}\right)-1\right)\left(\frac{1}{m_2}\right) \end{aligned} \tag{1}$$

In general, the forward rates at time 0 for the period spanning time *w* to time *z* is defined as

$$F_{0,w,z} = \left(\left(\frac{1+L_{0,z}h_{0,z}}{1+L_{0,w}h_{0,w}}\right)-1\right)\left(\frac{1}{m_z}\right) \tag{2}$$

It should be noted that the spot rate $L_{0,1}$ can be shown to be equivalent to a forward rate, specifically $F_{0,0,1}$. Using Equation (2), we have[2]

[2]In the proof below, there is a rate $L_{0,0}$ and a time factor $h_{0,0}$. The rate is the spot rate for interest from the period 0 to 0, which is obviously zero. The time factor is the number of days from time 0 time 0 divided by 360, which is also zero.

$$F_{0,0,1} = \left(\left(\frac{1+L_{0,1}h_{0,1}}{1+L_{0,0}h_{0,0}}\right)-1\right)\left(\frac{1}{m_1}\right)$$
$$= \left(1+L_{0,1}h_{0,1}-1\right)\left(\frac{1}{m_1}\right)$$
$$= L_{0,1}h_{0,1}\left(\frac{1}{m_1}\right) = L_{0,1}$$

In pricing the swap, we need to find the fixed rate such that the present value of the fixed payments equals the present value of the floating payments. That rate, *R*, is defined as

$$\sum_{t=1}^{T}\left(F_{0,t-1,t}-R\right)m_t\left(\frac{1}{1+L_{0,t}h_{0,t}}\right)=0 \tag{3}$$

We restate this to the following:

$$\sum_{t=1}^{T}F_{0,t-1,t}m_t\left(\frac{1}{1+L_{0,t}h_{0,t}}\right)=R\left(\sum_{t=1}^{T}m_t\left(\frac{1}{1+L_{0,t}h_{0,t}}\right)\right)$$

And the solution is

$$R=\frac{\sum_{t=1}^{T}F_{0,t-1,t}m_t\left(\frac{1}{1+L_{0,t}h_{0,t}}\right)}{\sum_{t=1}^{T}m_t\left(\frac{1}{1+L_{0,t}h_{0,t}}\right)} \tag{4}$$

**Pricing Swaps as Combinations of Bonds**

The alternative approach to swap pricing is to treat the swap as a combination of bonds. For a pay-fixed swap, we assume that we issue a fixed-rate bond at the rate *R* and use the proceeds to buy a floating-rate bond paying LIBOR. The payment dates and specifications on the bonds must be exactly as they are on a swap. Of course, bonds have principal payments and swaps do not, but since we are issuing a bond and using the proceeds from the principal to buy another bond, the effects offset at the start. At maturity, we use the proceeds from the bond we own to repay the principal of the bond we issued, so those principals again offset. Thus, we can treat swaps like a pair of bonds, as the pair of bonds will have the same cash payments.[3]

It is well-known and easy to prove that a floating-rate bond in which the coupon is the same rate as the discount rate will have a value of par on each payment date and at the start. Thus, the present value of the floating-rate bond payments is 1. We then specify a fixed-rate bond with rate *R* in which the value of the bond is 1.

[3]We are, of course, assuming away any credit risk.

$$1=\sum_{t=1}^{T}Rm_t\left(\frac{1}{1+L_{o,t}h_{0,t}}\right)+\frac{1}{1+L_{0,T}h_{0,T}}$$

We then solve for *R* to obtain

$$R=\frac{1-\left(\frac{1}{1+L_{0,T}h_{0,T}}\right)}{\sum_{t=1}^{T}m_t\left(\frac{1}{1+L_{o,t}h_{0,t}}\right)} \tag{5}$$

**Equivalence of the Two Approaches**

What we want to show is how Equations (4) and (5) are equivalent. The denominators are already equivalent, so we can confine our focus to the numerators. Let us write out the numerator of Equation (4),

$$\begin{aligned}&\sum_{t=1}^{T}F_{0,t-1,t}m_t\left(\frac{1}{1+L_{0,t}h_{0,t}}\right)\\&=F_{0,0,1}m_1\frac{1}{1+L_{0,1}h_{0,1}}+F_{0,1,2}m_2\frac{1}{1+L_{0,2}h_{0,2}}+...+F_{0,T-1,T}m_T\frac{1}{1+L_{0,T}h_{0,T}}\\&=L_{0,1}m_1\frac{1}{1+L_{0,1}h_{0,1}}+F_{0,1,2}m_2\frac{1}{1+L_{0,2}h_{0,2}}+...+F_{0,T-1,T}m_T\frac{1}{1+L_{0,T}h_{0,T}}\end{aligned}$$

Next, we substitute for the forward rates using Equation (2),

$$\begin{aligned}&L_{0,1}m_1\frac{1}{1+L_{0,1}h_{0,1}}+\left(\frac{1+L_{0,2}h_{0,2}}{1+L_{0,1}h_{0,1}}-1\right)\left(\frac{1}{m_2}\right)m_2\frac{1}{1+L_{0,2}h_{0,2}}\\&+...+\left(\frac{1+L_{0,T}h_{0,T}}{1+L_{0,T-1}h_{0,T-1}}-1\right)\left(\frac{1}{m_T}\right)m_T\frac{1}{1+L_{0,T}h_{0,T}}\end{aligned}$$

And this result can be written as

$$\begin{aligned}&L_{0,1}m_1\frac{1}{1+L_{0,1}h_{0,1}}\\&+\left(\frac{1+L_{0,2}h_{0,2}-(1+L_{0,1}h_{0,1})}{(1+L_{0,1}h_{0,1})(1+L_{0,2}h_{0,2})}\right)+...+\left(\frac{1+L_{0,T}h_{0,T}-(1+L_{0,T-1}h_{0,T-1})}{(1+L_{0,T-1}h_{0,T-1})(1+L_{0,T}h_{0,T})}\right)\end{aligned}$$

Simplifying a bit, we obtain

$$L_{0,1}m_1\frac{1}{1+L_{0,1}h_{0,1}}$$

$$+\frac{1+L_{0,2}h_{0,2}}{(1+L_{0,1}h_{0,1})(1+L_{0,2}h_{0,2})}-\frac{1+L_{0,1}h_{0,1}}{(1+L_{0,1}h_{0,1})(1+L_{0,2}h_{0,2})}$$

$$+...+\frac{1+L_{0,T}h_{0,T}}{(1+L_{0,T-1}h_{0,T-1})(1+L_{0,T}h_{0,T})}-\frac{1+L_{0,T-1}h_{0,T-1}}{(1+L_{0,T-1}h_{0,T-1})(1+L_{0,T}h_{0,T})}$$

We can now see that most of the terms cancel,

$$L_{0,1}m_1\frac{1}{1+L_{0,1}h_{0,1}}+\frac{1}{1+L_{0,1}h_{0,1}}-\frac{1}{1+L_{0,2}h_{0,2}}+\frac{1}{1+L_{0,2}h_{0,2}}-\frac{1}{1+L_{0,3}h_{0,3}}...\frac{1}{1+L_{0,T-1}h_{0,T-1}}-\frac{1}{1+L_{0,T}h_{0,T}}$$

$$=\frac{L_{0,1}m_1}{1+L_{0,1}h_{0,1}}+\frac{1}{1+L_{0,1}h_{0,1}}+...-\frac{1}{1+L_{0,T}h_{0,T}}$$

$$=\frac{1+L_{0,1}m_1}{1+L_{0,1}h_{0,1}}-\frac{1}{1+L_{0,T}h_{0,T}}$$

$$=1-\frac{1}{1+L_{0,T}h_{0,T}}$$

Note that we use the fact that $m_1 = h_{0,1}$. With equivalence of the numerators and the same denominator, this completes the proof that the forward rate approach equals the bonds approach.