TEACHING NOTE 18-02:
PRICING INTEREST RATE SWAPS WITH ROLLOVER FLOATING RATES

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In the aftermath of the financial crisis of 2008-2009 and the scandal in which some banks apparently manipulated LIBOR, interest rate swap pricing has evolved into a new phase. Of course, one of the primary issues is the fact that historically swaps have been treated as having no default possibility. After the financial crisis, the instability of the banking system meant that the dealers were not default free. We shall set that issue aside for this note, however, in order to address this new phase in the evolution of swap pricing: the use of a rollover floating rate to price swaps. Oftentimes, this rollover floating rate will be the Overnight Indexed Swap rate instead of LIBOR, but what follows here applies to any type of swap in which the rate is a rollover rate.

What we mean by this concept is fairly simple, but to understand it, let us remind ourselves how a standard vanilla swap would be constructed. Consider a two-year swap based on 180-day LIBOR. Such a swap will have semi-annual payments approximately 180 days apart. In most cases, the swap will be structured with the assumption of the actual day count between payment dates with a denominator of 360. For example, if the swap starts on March 15 and the first payment is on September 15, there are 16 + 30 + 31 + 30 + 31 + 31 + 15 = 184 days in that period, March 15 to September 15. Thus, the annual rate will be multiplied by 184/360 to determine the payment. As we know, in some countries and for some rates, such a period would be assumed to be 180 days and 365 days could be used in the denominator. Here we shall always use the actual day count and 360 days, a convention referred to as ACT/360. So, on March 15, we observe six-month LIBOR, technically 180-day LIBOR, and that rate determines the floating payment that will occur on September 15. On September 15, the net of the fixed and floating payments is made, and a new rate is observed that determines the floating payment on the following March 15.

In an increasing number of cases in today's world, the floating payment on a swap is starting to be specified as a rollover-type rate. The OIS, or Overnight Indexed Swap rate, is indeed a common case. This rate is technically just the Fed funds rate, which is the rate at which banks borrow and lend excess reserves to and from each other.[1] The loan is for one day only, and the borrowing bank repays the money the next day. It might then take out another loan for another day. Or, if it has excess reserves, it might lend the money overnight. These loans have no collateral, and the interest is added on to the principal by multiplying the rate by the factor, rate/360. So, for example, if a bank borrows \$1 for five straight days at the daily rates r_0, r_1, r_2, r_3, and r_4, it will pay back,[2]

$$\$1\left(1+\frac{r_0}{360}\right)\left(1+\frac{r_1}{360}\right)\left(1+\frac{r_2}{360}\right)\left(1+\frac{r_3}{360}\right)\left(1+\frac{r_4}{360}\right).$$

Sometimes companies take out loans in which the floating rate they pay is a compounded daily rate. For example, consider a one-month loan in which the borrower receives \$1 today, say June 15 and pays back the principal and accumulated interest on July 15. On July 15, the interest is determined by the compound factor reflecting the rolled over daily rates from June 15 to July 15. Of course, some of those days are Fridays, so the rate on Fridays is also the rate for Saturday and Sunday. Thus, that factor $(1 + r/360)$ for a Friday would be effectively raised to the third power. Similar adjustments are done for holidays.

Just to keep things simple so we can illustrate the point without too many terms, suppose a loan is taken out on a Monday at the rate r_{Mon}. On Tuesday, the rate adjusts to r_{TUE}, and so forth. This process continues until the loan is repaid the following Monday. The amount that would be repaid is, thus,

$$\$1\left(1+\frac{r_{MON}}{360}\right)\left(1+\frac{r_{TUE}}{360}\right)\left(1+\frac{r_{WED}}{360}\right)\left(1+\frac{r_{THU}}{360}\right)\left(1+\frac{r_{FRI}}{360}\right)^3$$

In other words, this pattern would be followed to accumulate the interest to reflect the compounding of the daily rates.

Because swaps are often used to hedge loans, a swap to hedge a floating-rate loan in which the rates are rolled over would likely be structured to pay off based on the compounding of the daily fed funds rate. We now determine how the pricing of these swaps differs from pricing standard swaps, wherein the floating rate is set at the beginning of the period and applies to the entire settlement period, with the floating interest payment made at the end of the period.

We shall now develop the theory of standard swap pricing, followed by an example. Then we shall do swap pricing when the underlying rate rolls over, as explained above.

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[1]In the European market, the analogous rate is called the EONIA, which stands for Euro Overnight Index Average, and in the U. K., the analogous rate is called SONIA, which stands for Sterling Overnight Index Average.

[2]In the formula below, there is an implicit 1/360, but the “1” is simply dropped.

Notation

Suppose we are interested in a financial instrument, such as a bond or a swap, with n = 1, …, N payments at various dates. The number of days to each respective payment date is $h_{0,1}$, $h_{0,2}$, …, $h_{0,N}$. We shall also require that the current date, time 0, be specified as $h_{0,0}$, which equals 0. The number of days between payment dates is $h_{0,1}$ for the first payment, $h_{0,2} - h_{0,1}$ for the second payment, and so on and for the last payment is $h_{0,N} - h_{0,N-1}$.

Let us start by assuming that we have full information available on the term structure for each of these dates, meaning that we know the rates for the given dates on which the swap payments will occur. Using L, as a general symbol for the rate, this means we know the rates $L_{0,1}$, $L_{0,2}$, …, $L_{0,N}$.[3] The subscript 0,1 for example means that this rate is LIBOR observed at time 0 for the period spanning to the payment date of time 1. Thus, $L_{0,n}$ is $h_{0,n}$-day LIBOR. Given the rate and number of days, let $DF_{0,1}$ be the discount factor corresponding to the rate $L_{0,1}$ and number of days $h_{0,1}$, and is given by the following general expression for any period, 0,n.

$$DF_{0,n} = \frac{1}{1 + L_{0,n}\left(\frac{h_{0,n}}{360}\right)} \tag{1}$$

We can now proceed to price swaps.

Standard Swaps

To repeat a point made earlier, note that the fixed payments on a swap can be determined using an actual day count since the last payment or with a standard assumption of 30 days in a month. The number of days assumed in a year can be either 360 or 365. It is customary for swaps involving dollars, however, that floating payments based on LIBOR always use an actual day count and 360 days in a year.[4] Thus, swaps can be based on different day counts on the fixed and floating legs, but they do generally use the same annual day count assumption. This assumption can seem very confusing, but it is easy to reconcile the different approaches using simple principles of time value of money. For our purposes, we shall use ACT/360.

Standard Swap Pricing with Full Information: The Theory

Pricing the swap means to find the fixed rate when the contract is written. To do so, recall that a swap can be viewed as the issuance of either a floating-rate bond or a fixed-rate bond with the proceeds used to buy the other bond. Interest rate swap payments themselves do not involve payment of the notional principal either at the start or at the maturity date, but by issuing

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[3] L is chosen to stand for LIBOR, but other rates and symbols could be used and will when we get to OIS swaps.

[4] Floating payments based on the British pound, Australian dollar, New Zealand dollar, and Hong Kong dollar use 365 days. All other countries tend to use 360 days.

one bond and using the proceeds to buy another, the principals offset, so that the cash flows on the bond issued and bond purchased correspond to the cash payments on a swap. Without loss of generality, let us assume that we issue a fixed-rate bond and use the proceeds to buy a floating-rate bond. The coupon on the fixed-rate bond will correspond to the coupon on the swap. We start by assuming this bond has N payments with the N^{th} payment being the last interest payment and the principal.

Now we also need a symbol to represent the number of days in each respective period. The symbol $h_{0,n}$ was specified as the cumulative number of days from day 0 to day n, on which the n^{th} payment occurs. Therefore, we can find the number of days between payments by subtraction. Recall that we specified the current day as $h_{0,0} = 0$. Thus, the number of days to the first payment is $h_{0,1} - h_{0,0} = h_{0,1}$. The number of days from the first payment to the second is $h_{0,2} - h_{0,1}$. The number of days from the next-to-last payment to the last is $h_{0,N} - h_{0,N-1}$.

Suppose we are interested in a swap with N payments. Let us assume we can find a bond whose payment dates correspond to those of the swap and let the bond coupon represent the swap fixed rate, which we designate as $R_{0,N}$, meaning the fixed rate on a swap starting at time 0 and having N payments. The payments are at times $n = 1, 2, \ldots, N$ payments at various dates as indicated by $h_{0,1}, h_{0,2}, \ldots, h_{0,N}$. Assuming a \$1 par value, the price or present value of a bond whose coupon rate is the swap fixed rate, $R_{0,N}$ is

$$B_{0,N} = \sum_{n=1}^{N} R_{0,N}\left(\frac{h_{0,n} - h_{0,n-1}}{360}\right)DF_{0,n} + DF_{0,N} \tag{2}$$

Setting this formula equal to the present value of the floating payments plus the hypothetical notional of 1, we have

$$\sum_{n=1}^{N} R_{0,N}\left(\frac{h_{0,n} - h_{0,n-1}}{360}\right)DF_{0,n} + DF_{0,N} = 1 \tag{3}$$

Solving for $R_{0,N}$, we have

$$R_{0,N} = \frac{1 - DF_{0,N}}{\sum_{n=1}^{N}\left(\frac{h_{0,n} - h_{0,n-1}}{360}\right)DF_{0,n}} \tag{4}$$

Note how the discount factors have to be weighted by the day-count adjustment. This multiplication expresses the solution $R_{0,N}$ being stated on an annualized basis, as it always is.

Standard Swap Pricing: Example

Now let us work a problem. Shown below is the term structure information we have and shall need to value a six-month swap with two monthly payments in 31 and 61 days, respectively.

Table 1. Term Structure for Two Monthly Payments

	Cumulative			Days	
	Days	Rate		per period	
Payment	$h_{0,n}$	$R_{0,n}$	$DF_{0,n}$	$h_{0,n}$ - $h_{0,n-1}$	($h_{0,n}$ - $h_{0,n-1}$)/360
1	31	0.0440	0.9962	31	0.0861
2	61	0.0450	0.9924	30	0.0833

Solving for the rate using Equation (4) gives

$$R_{0,2} = \frac{1-0.9924}{0.0861(0.9962)+0.0833(0.9924)}$$
$$= 0.0449$$

Thus, the rate is 4.49%.

Swaps with a Rollover Floating Rat

To price a swap with a rollover floating rate, let us first make a change of symbols. Let us use $f_{0,n}$ instead of $R_{0,n}$, with the f representing the Fed Funds rate. We assume that there are j = 1, .., J rates in each settlement period such that these rates compound to determine the floating rate on the swap. Using the example in Table 1, the first payment occurs in 31 days. Suppose there is a separate floating rate for each of the 31 days. On Day 31, the compounded floating rate determines the rate that will apply to the first swap payment. In other words, the floating swap payment for a set of 31 compounded rates, which we shall call f_1, f_2, …, f_{31}, would be

$$\left(1+\frac{f_1}{360}\right)\left(1+\frac{f_2}{360}\right)\ldots\left(1+\frac{f_{31}}{360}\right)-1$$

As noted earlier, for the rate on a Friday, we simply treat that rate as the rate for Saturday and Sunday.

Now let us take an abbreviated example. Suppose the swap starts at the end of the day on Wednesday and lasts one week. Table 2 below shows how the interest is calculated.

Table 2. Example of Interest on Seven-day Loan or Swap

Day #	Day	Overnight rate	Compound factor	Cumulative Interest
0	Wednesday	0.0225		
1	Thursday	0.0228	1.00006250	1.00006250
2	Friday	0.0227	1.00006333	1.00012584
3	Saturday	0.0227	1.00006306	1.00018890
4	Sunday	0.0227	1.00006306	1.00025197
5	Monday	0.0231	1.00006306	1.00031504
6	Tuesday	0.0232	1.00006417	1.00037923

7	Wednesday	NA	1.00006444	1.00044370

Thus, at the end of the day on the first Wednesday, the overnight rate is 2.25%. The compound factor for the next day is, therefore,

$$1+\frac{0.0225}{360}=1.00006250$$

The cumulative interest per dollar is shown in the last column. So, by the end of the day on Thursday, the loan has a cumulative payoff of 1.00006250. The overnight rate for Thursday night is 2.28%, so the compound factor is

$$1+\frac{0.0228}{360}=1.00006333$$

And as of the end of the day on Friday, the cumulative value of the loan

$$(1.00006250)(1.00006333)=1.00012584\ ,$$

which is shown in the right column. Note that the overnight rate for Friday, 2.27%, also becomes the overnight rate for Saturday and Sunday, and this rate is not reset until Monday at the end of the day. As of the end of the following Wednesday, we see that the cumulative value of the loan is 1.00044370. Thus, the interest payment is 0.00044370. For one week, this equates to an annual rate of

$$0.00044370(52)=0.02307216$$

Thus, 2.307216% was the effective compound rate over the week.

In order to understand swap pricing when the floating rate rolls over, we must first take a look at floating-rate loans and securities when the rate rolls over. In pricing a standard swap, we have already taken advantage of the fact that, credit considerations aside, on each reset date, the value of the security returns to its par value of 1. For a rollover floating-rate note, that statement is true on the payment date, but not on any other date.

We can illustrate this point with a very simple example. Consider a floating-rate note with two payment dates. At each payment date, the interest paid is the compound interest factor over three sub-periods. A practical example, might be a six-month swap with two payments that occur after the first three months and after the second three months. Each payment is based on a compounded monthly rate. Thus, the first payment will reflect the first month's rate, compounded by the second month's rate, then compounded by the third month's rate.

Let r_{01} be the one-period rate observed at time 0, r_{12} be the one-period rate observed at time 1, r_{23} be the one-period rate observed at time 2, etc. With three sub-periods over which interest is compounded and two interest payment dates, there are six periods in total.

The first interest payment will be at the rate

$$\left(1+\frac{r_{01}}{360}\right)\left(1+\frac{r_{12}}{360}\right)\left(1+\frac{r_{23}}{360}\right)-1$$

And the second interest payment will be at the rate

$$\left(1+\frac{r_{34}}{360}\right)\left(1+\frac{r_{45}}{360}\right)\left(1+\frac{r_{56}}{360}\right)-1$$

When this payment is made, the principal will also be repaid. So, at the maturity date, the total payment will be the equation directly above plus 1.

Now, let us step back to the fifth date and value the floating-rate note. It is the present value of the upcoming payment of 1 plus the interest payment as shown in the equation directly above,

$$\frac{\left(1+\frac{r_{34}}{360}\right)\left(1+\frac{r_{45}}{360}\right)\left(1+\frac{r_{56}}{360}\right)}{\left(1+\frac{r_{56}}{360}\right)}=\left(1+\frac{r_{34}}{360}\right)\left(1+\frac{r_{45}}{360}\right)$$

Note that we discount at r_{56} as that is the one-period rate at time 5. Now, let us step back one period further to time 4 and value the floating-rate note by discounting its value at time 4,

$$\frac{\left(1+\frac{r_{34}}{360}\right)\left(1+\frac{r_{45}}{360}\right)}{\left(1+\frac{r_{45}}{360}\right)}=\left(1+\frac{r_{34}}{360}\right)$$

And note that we discount at r_{45} as that is the one-period rate at time 4. Next, let us step back one period further to time 3 and value the floating rate note by discounting the value at time 4, which is directly above,

$$\frac{\left(1+\frac{r_{34}}{360}\right)}{\left(1+\frac{r_{34}}{360}\right)}=1$$

Note that at time 3, which is a payment date, the floating-rate note returns to its par value of 1. Now observed that the payment at time 3 will be

$$\left(1+\frac{r_{01}}{360}\right)\left(1+\frac{r_{12}}{360}\right)\left(1+\frac{r_{23}}{360}\right)-1$$

When we step back to time 2, we value the payment at time 3 plus the value of the remaining payments, which we just said is equal to 1. Thus, the value at time 2 is

$$\frac{\left(\left(1+\frac{r_{01}}{360}\right)\left(1+\frac{r_{12}}{360}\right)\left(1+\frac{r_{23}}{360}\right)-1\right)+1}{\left(1+\frac{r_{23}}{360}\right)}=\left(1+\frac{r_{01}}{360}\right)\left(1+\frac{r_{12}}{360}\right)$$

Now step back to time 1 and value the upcoming value as

$$\frac{\left(1+\frac{r_{01}}{360}\right)\left(1+\frac{r_{12}}{360}\right)}{\left(1+\frac{r_{12}}{360}\right)}=\left(1+\frac{r_{01}}{360}\right)$$

And finally, we step back to time 0 and value the floating-rate note as

$$\frac{\left(1+\frac{r_{01}}{360}\right)}{\left(1+\frac{r_{01}}{360}\right)}=1$$

And of course, the floating-rate note returns to a value of 1.

What we have just seen is that a floating-rate note is valued at par on the initial date and on each payment date. On the intermediate reset dates, it is valued at par plus the accumulated interest since the last reset date. This useful result tells us that if we were pricing a swap, we could add a hypothetical notional to both the fixed and floating sides and be assured that the floating side is valued at 1 at the start.

Now, to price the swap, we have nothing to do that we have not already done. The present value of the fixed payments plus hypothetical notional is Equation (2), the present value of the floating payments plus hypothetical notional is 1. Setting the one equal to the other gives the swap rate exactly as specified in Equation (4).

We are not saying, however, that the swap fixed rate would be the same if priced off of the OIS or federal funds curve in comparison to the LIBOR curve. The curves are different. Moreover, we are not saying that it is easy to price off the OIS curve since that curve truly only exists for very short maturities. What we are saying is that given the OIS curve, the pricing is the same. In other words, swap pricing when floating rates roll over is the same.

Let us do an example. Recall back in Table 1, we had a two-month swap in which the floating rate is a rolled over rate for 31 days in the first period and 30 in the second. At the time the swap is set up, the 31-day rate is 4.40% and the 61-day rate is 4.50%. The calculations are the same as in Table 1. Of course, the difficulty is that there have to be a 31- and 61-day rates available. If there are not, these rates would need to be inferred as a premium over another set of rates, such as the t-bill or LIBOR.

Valuation of Swaps with Rollover Floating Rates

Valuation of a swap with a rollover floating rate will differ somewhat from the valuation of a standard swap. In a standard swap, the next floating payment is known because the floating rate that determines the next payment has been set at the previous settlement date. The floating payments that are beyond the next payment date are not known but their value as of the next payment date is equal to par value of 1. Thus, one can find the present value of the floating payments plus hypothetical notional of 1 as the value of the next floating payment plus 1 discounted back to the valuation date. The value of the fixed payments plus notional of 1 is also discounted back to the valuation date.

For a swap with a floating rollover rate, valuing the floating payments is somewhat different. We have to take into account that we do not know the next floating payment, but we do know the accumulated compound value of the floating payments since the last settlement date. Let us go back to our example of a swap with two payment dates and three rates within each payment period that compound to equal the floating rate paid at the end of the period. Recall that the sequence of rates were denoted as r_{01}, r_{12}, r_{23}, r_{34}, r_{45}, and r_{56}. Table 3 shows essentially what we already showed above, the valuation of a floating-rate instrument whose payments are determined by rolling over the floating rates.

Table 3. Valuing a Floating Rate Note with Rollover Floating Payments

Time (t)	Payment	Value at Time t
0		$\frac{(1+r_{01})}{1+r_{01}}=1$
1		$\frac{(1+r_{01})(1+r_{12})}{1+r_{12}}=1+r_{01}$
2		$\frac{[(1+r_{01})(1+r_{12})(1+r_{23})-1]+1}{1+r_{23}}=(1+r_{01})(1+r_{12})$
3	$(1+r_{01})(1+r_{12})(1+r_{23})-1$	$\frac{1+r_{34}}{1+r_{34}}=1$
4		$\frac{(1+r_{34})(1+r_{45})}{1+r_{45}}=1+r_{34}$
5		$\frac{(1+r_{34})(1+r_{45})(1+r_{56})}{1+r_{56}}=(1+r_{34})(1+r_{45})$

6	$[(1+r_{34})(1+r_{45})(1+r_{56})-1]+1$	$(1+r_{34})(1+r_{45})(1+r_{56})$

What we see here is that the value on any payment date and at the start is 1. The value at the intervening dates is the value of the principal plus accumulated the interest. The value of the fixed payment is easily obtained as the value of the remaining fixed payments plus hypothetical notional paid off at maturity. We simply have to know the discount rates for the period from the valuation date to the next payment date. The swap value is then the difference in these fixed and floating payments.