TEACHING NOTE 18-01:
PRICING INTEREST RATE SWAPS WITH LIMITED TERM STRUCTURE INFORMATION

Version date: May 25, 2018 C:\Users\Don Chance\Documents\Professional Folders\Class Material\Teaching Notes\2018 TNs\TN18-01.docx

In the absence of credit considerations, interest rate swap pricing is relatively simple. An interest rate swap can be decomposed into a fixed-rate bond and an opposite position in a floating-rate bond. If the floating payments are determined by the evolution of a floating rate, and that rate is appropriate for discounting those payments, then a floating-rate bond always has a value of par at the issue date or a payment date. One can then easily solve for the rate on a fixed-rate bond that equates its value to par.

For whatever floating rate is appropriate for the period of interest, a term structure of spot rates must be known for maturities corresponding to the dates on which the floating payments will be made. For example, if LIBOR is used, pricing a swap would require the LIBOR term structure over a full range of maturities that span the life of the swap. We would need to know LIBOR for the period from today until each swap payment date. For valuing the swap during its life, say between payment dates, one must know the LIBOR or any other rate is being used for each period from the valuation date to each of the payment dates.

At the time of this writing, LIBOR is quoted for maturities of overnight, one week, two weeks, and 1-12 months for the dollar, pound, euro, yen, and Swiss franc. Certain currencies have fewer quoted rates available.[1] If a rate exists that corresponds to the period necessary to value the swap, then the LIBOR spot rates can be used. Unfortunately, many swaps have maturities of more than a year. In addition, some maturities and payment dates will not align with the maturities corresponding to available rates. This note addresses these types of special cases. We shall disregard credit issues, which are covered in another note.

Notation

Suppose we are interested in a financial instrument, such as a bond or a swap, with $n = 1, \ldots, N$ payments at various dates. The number of days to each respective payment date is $h_{0,1}$, $h_{0,2}$, …, $h_{0,N}$. We shall also require that the current date, time 0, be specified as $h_{0,0}$, which equals 0.

[1]A good source of rates is www.global-rates.com/interest-rates/libor/libor.aspx

The time between payment dates is $h_{0,1}$ for the first payment, $h_{0,2} - h_{0,1}$ for the second payment, and so on to the last payment, $h_{0,N} - h_{0,N-1}$.

Let us start by assuming we have full information available on the term structure for each of these dates. Using L, as the rate, this means we know the rates $L_{0,1}$, $L_{0,2}$, …, $L_{0,N}$.[2] The subscript 0,1 for example means that this is LIBOR observed at time 0 for the period spanning to the payment date of time 1. Thus, $L_{0,n}$ is $h_{0,n}$-day LIBOR. Given the rate and number of days, let $DF_{0,1}$ be the discount factor corresponding to the rate $L_{0,1}$ and number of days $h_{0,1}$, and it equals the following,

$$DF_{0,n} = \frac{1}{1 + L_{0,n}\left(\frac{h_{0,n}}{360}\right)} \tag{1}$$

We can now proceed to price swaps. We will start by reviewing the procedure under the assumption that there is no information missing.

Pricing Swaps when the Full Term Structure is Available

So we first assume we have information on the full term structure. This means that we know either the rate or the discount factor for all dates corresponding to the payment dates on the swap. First note that the fixed payments on a swap can be determined using an actual day count since the last payment or with a standard assumption of 30 days in a month. The number of days assumed in a year can be either 360 or 365. It is customary for swaps involving dollars, however, that floating payments based on LIBOR always use an actual day count and 360 days in a year.[3] Thus, swaps can be based on different day counts on the fixed and floating legs, but they do generally use the same annual day count assumption. This assumption can seem very confusing, but it is easy to reconcile the different approaches using simple principles of time value of money. We shall not digress into this point right now, but see the Appendix for an explanation.

Standard Swap Pricing with Full Information: The Theory

Pricing the swap means to find the fixed rate. To do so, recall that a swap can be viewed as the issuance of either a floating-rate bond or a fixed-rate bond with the proceeds used to buy the other bond. Swap payments themselves do not involve payment of the notional principal either at the start or at the maturity date, but by issuing one bond and using the proceeds to buy another, the principals offset, so that the cash flows on the bond issue/bond purchase correspond to the cash payments on a swap. Without loss of generality, let us assume we that issue a fixed-

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[2] L is chosen to stand for LIBOR, but other rates could be used.

[3] Floating payments based on the British pound, Australian dollar, New Zealand dollar, and Hong Kong dollar use 365 days. All other countries tend to use 360 days.

rate bond and use the proceeds to buy a floating-rate bond. The coupon on the fixed-rate bond will correspond to the coupon on the swap. We start by assuming this bond has N payments with the N^{th} payment being the last interest payment and the principal. We shall assume that the fixed payments are based on ACT/360, but we must keep in mind that other assumptions are possible and can be easily accommodated in what follows.

Now we also need a symbol to represent the number of days in each respective period. The symbol $h_{0,n}$ was specified as the cumulative number of days from day 0 to day n, on which the n^{th} payment occurs. Therefore, we can find the number of days between payments by subtraction. Recall that we specified the current day as $h_{0,0} = 0$. Thus, the number of days to the first payment is $h_{0,1} - h_{0,0} = h_{0,1}$. The number of days from the first payment to the second is $h_{0,2} - h_{0,1}$. The number of days from the next-to-last payment to the last is $h_{0,N} - h_{0,N\text{-}1}$.

Suppose we are interested in a swap with N payments. Assume we find a bond whose payment dates correspond to those of the swap and let the bond coupon represent the swap fixed rate, which we designate as $R_{0,N}$, meaning the fixed rate on a swap starting at time 0 and having N payments. The payments are at times $n = 1, 2, \ldots, N$ payments at various dates as indicated by $h_{0,1}, h_{0,2}, \ldots, h_{0,N}$. Assuming a \$1 par value, the price or present value of a bond whose coupon rate is the swap fixed rate, $R_{0,N}$ is

$$B_{0,N} = \sum_{n=1}^{N} R_{0,N}\left(\frac{h_{0,n} - h_{0,n-1}}{360}\right)DF_{0,n} + DF_{0,N} \tag{2}$$

Setting this formula equal to the present value of the floating payments of 1, we have

$$\sum_{n=1}^{N} R_{0,N}\left(\frac{h_{0,n} - h_{0,n-1}}{360}\right)DF_{0,n} + DF_{0,N} = 1 \tag{3}$$

Solving for $R_{0,N}$, we have

$$R_{0,N} = \frac{1 - DF_{0,N}}{\sum_{n=1}^{N}\left(\frac{h_{0,n} - h_{0,n-1}}{360}\right)DF_{0,n}} \tag{4}$$

Note how the discount factors have to be weighted by the day-count adjustment. This multiplication puts the solution $R_{0,N}$ being stated on an annualized basis, as it always is.

Standard Swap Pricing: Example

Let us work a problem. Shown below is the term structure information we have and shall need to value a two-year swap with four semi-annual payments in 182, 365, 547, and 730 days, respectively.

Table 1. Term Structure for Four Semi-Annual Payments

	Cumulative			Days	
Payment	Days	LIBOR		per period	
n	$h_{0,n}$	$L_{0,n}$	$DF_{0,n}$	$h_{0,n}$ - $h_{0,n-1}$	($h_{0,n}$ - $h_{0,n-1}$)/360
1	182	2.50%	0.9875	182	0.5056
2	365	2.80%	0.9724	183	0.5083
3	547	2.90%	0.9578	182	0.5056
4	730	2.95%	0.9436	183	0.5083

Solving for the rate using Equation (4) gives

$$R_{0,4} = \frac{1-0.9436}{0.5056(0.9875)+0.5083(0.9724)+0.5056(0.9578)+0.0.5083(0.9436)}$$
$$= 0.0288$$

And the rate is 2.88%.

Bootstrapping the Term Structure from the Bond Market

Now suppose there is no term structure to provide the rate for a given maturity. For example, suppose we need to price a two-year swap, but we have LIBOR for only one year, which is the actual limit for LIBOR. In that situation and in others where rates are not publicly quoted, we can sometimes infer the discount rate from the observable price of a coupon bond. This method is called *bootstrapping*. In bootstrapping the term structure, one assumes that bond investors are able to either determine the unspecified pieces of the term structure or that they price the bonds to the best of their ability without that information. We can use the information from such a bond to infer a single missing rate. For example, if we have LIBOR only for one year, but have a two-year swap to price, we need two additional pieces of the term structure. Using Table 1, this situation would be like having the first two rates, 2.50% and 2.80% but not knowing the last two. If we can find a bond that makes three payments in total and the last payment occurs on the 547th day from today, we can infer what rate was used to discount that payment. We shall address the case when the bond maturity does not line up with the desired point on the term structure. Specifically, there will be gaps to fill in.

Bootstrapping from Bond Prices: The Theory

So let us say that we have knowledge of only Q payment points on the term structure, and we need to price a swap with N = Q + 1 > Q payments. We find a bond with Q + 1 payments where payment Q + 1 occurs on the date of the swap payment. We observe the price of that bond and infer the Q + 1 discount factor that would be consistent with the observable price of the bond. For example, let $B_{0,Q+1}$ be the observable price of the bond, and let $c_{0,Q+1}$ be the coupon rate of that bond. The observable price is, by definition, the present value of the payments as follows:

$$B_{0,Q+1} = \sum_{n=1}^{Q+1} c\left(\frac{h_{0,n} - h_{0,n-1}}{360}\right)DF_{0,n} + DF_{0,Q+1} \tag{5}$$

Let us separate the payments to isolate the payment at day Q + 1:

$$B_{0,Q+1} = \sum_{n=1}^{Q} c_{0,Q+1}\left(\frac{h_{0,n} - h_{0,n-1}}{360}\right)DF_{0,n} + c_{0,Q+1}\left(\frac{h_{0,Q+1} - h_{0,Q}}{360}\right)DF_{0,Q+1} + DF_{0,Q+1} \tag{6}$$

The terms associated with the summation sign are the ones based on the observable term structure rates. The remaining terms are associated with the unknown rate $L_{0,Q+1}$. We need to solve for $DF_{0,Q+1}$. All of the remaining information in Equation (6) is known. Isolating $DF_{0,Q+1}$, we have

$$DF_{0,Q+1} = \frac{B_{0,Q+1} - \sum_{n=1}^{Q} c_{0,Q+1}\left(\frac{h_{0,n} - h_{0,n-1}}{360}\right)DF_{0,n}}{\left(1 + c_{0,Q+1}\left(\frac{h_{0,Q+1} - h_{0,Q}}{360}\right)\right)} \tag{7}$$

Once we have the discount factor, the implied interest rate is easily found by turning around Equation (1),

$$L_{0,N} = \left(\frac{1}{DF_{0,n}} - 1\right)\frac{360}{h_{0,n}} \tag{8}$$

Bootstrapping the Term Structure from the Bond Market: Example

Now let us work the example using the term structure in Table 1. We will assume we have the first two rates, but in keeping with the characteristics of the LIBOR market, we assume we do not know the rates for more than one year. Assume we want to solve for the rate for 547 days. If successful, we should obtain a rate of 2.90%. First we observe in the market a bond that makes its last payment 547 days from now. Let us assume its coupon is 3.25%, and its price observed in the market is 1.0045. Using Equation (7), we have

$$DF_{0,Q+1} = \frac{1.0058 - 0.0325(0.5056(0.9875) + 0.5083(0.9724)}{1 + 0.0326(0.5056)} = 0.9578$$

This is the discount factor for a 547-day payment, as shown in the table above. We then convert it into a rate using Equation (8). Letting $n = Q$+1 and noting that $h_{0,Q+1}$ = 547, we have

$$L_{0,N} = \left(\frac{1}{0.9578} - 1\right)\frac{360}{547} = 0.0290$$

And this is the rate as given in the full term structure in Table 1. Continuing in this manner with additional bonds, we could extend the term structure further. Once we have the entire term structure filled out to the maturity date of the swap, we can price the swap.

Bootstrapping the Term Structure from the Swaps Market

Now let us price a swap when there is information missing that cannot be constructed from bootstrapping bond prices. Instead, we shall look to the swaps market. Again, consider a swap that has Q+1 payments, but we know LIBOR for only Q payments. Again, assuming away credit issues, the present value of the fixed rates on a swap should equal par value, which can be standardized to a value of 1. Letting $R_{0,Q+1}$ be the fixed rate on a swap with Q+1 payments with the last being on day h+1, we adapt Equation (3), which equates the present value of the fixed payments plus notional of 1 to the present value of the floating payments plus notional of 1 in the following manner.

$$\sum_{n=1}^{Q} R_{0,Q+1}\left(\frac{h_{0,n}-h_{0,n-1}}{360}\right)DF_{0,n} + R_{0,Q+1}\left(\frac{h_{0,Q+1}-h_{0,Q}}{360}\right)DF_{0,Q+1} + DF_{0,Q+1} = 1 \qquad (9)$$

Note that what we have done is summed up the value of the Q fixed payments and isolated the value of the last or Q+1-date payment plus notional. We can then solve for $DF_{0,Q+1}$ as follows:

$$DF_{0,Q+1} = \frac{1-\sum_{n=1}^{Q} R_{0,Q+1}\left(\frac{h_{0,n}-h_{0,n-1}}{360}\right)DF_{0,n}}{1+R_{0,Q+1}\left(\frac{h_{0,Q+1}-h_{0,Q}}{360}\right)} \qquad (10)$$

In this manner, we are bootstrapping the term structure from the swaps market.

Bootstrapping the Term Structure from the Swaps Market: Example

Let us now work a problem. Suppose that again we have only the first two rates on the term structure, and we wish to find the third. We do, however, observe a swap in the market that has payments on the dates desired, with the third occurring 547 days from now. We find that its rate is 2.88%. Let us now use Equation (10) to obtain the implied $DF_{0,Q+1}$.

$$DF_{0,3} = \frac{1-0.0288(0.5056(0.9875)+0.5083(0.9724))}{1+0.0288(0.5056))} = 0.9578$$

And this, of course, is the discount factor for the third payment as seen in Table 1. As we showed previously, this discount factor implies a rate of 2.90%. Assuming a swap is available that pays on the 730th day, we could continue in the same manner, and we would obtain a discount factor for the fourth payment of 0.9436 and a rate of 2.95%, as in Table 1.

Filling in the Gaps

Obviously bootstrapping limits us to the specific maturities on which the payments are made. It is quite possible that no bond exists with payments on the specific days of our interest. In that case, one would simply build as much of a term structure as one could and then would interpolate. For example, in the above table we know that the 365-day rate is 2.80% and the 547-

day rate is 2.90%. If we wanted to know a 400-day rate, perhaps because we have a swap or a bond with a payment 400 days from now, we would interpolate as follows:

$$0.0280 + (0.0290 - 0.0280)\left(\frac{400-365}{547-365}\right) = 0.0282$$

In other words, we are looking for a rate that falls between 2.80% and 2.90%, a span of 10 basis points. We start with 2.80% and add an adjustment that represents the fraction of 10 basis points that corresponds to the ratio of the number of days since the 400th day and the 365th day, which is 400 – 365 = 35, to the difference between 547 and 365, which is 182. In other words, the rates for periods sandwiched between two periods are considered linearly proportional to the number of days in the period. Of course that means that interpolation is a linear adjustment. There are more sophisticated methods for constructing the term structure that take into account its typical non-linearity.

A Bit of a Conundrum

In spite of being able to extract the unknown pieces of the term structure from the bond market and/or swaps market, we still have a puzzle. The bond market and swaps market could not do their pricing without the spot rates. Thus, if we use observed bond prices and swap rates to infer the term structure for periods beyond the availability of an observable term structure, we are putting the cart before the horse. The bond and swaps market had to use a rate to do its pricing. So, how could those markets use a rate that is unobservable?

Yet, this is precisely what happens in the options market when we estimate implied volatility. We take an observable option price and infer the volatility a market participant used to obtain that price. So what we are doing here is inferring a term structure that was used to price a bond or a swap. In fact, inferring a term structure is a much cleaner operation. The principle underlying the inference of a term structure process is arbitrage. We assume markets permit no arbitrage. The no-arbitrage condition is a very mild constraint. When we infer a volatility, however, we have to assume an option pricing model, such as Black-Scholes-Merton or some other model. The requirements for that model, or any model to hold, are much more onerous than simply requiring that arbitrage opportunities do not exist because they are exploited rapidly.

Appendix: Actual Days vs. 30 Days per Month and 360 vs. 365 days per Year

Suppose an investor expects to receive $100 in exactly 182 days. That investor believes that the money is worth $96.55 today. Here are the quoted rates for the four assumptions on which short-term interest rates are commonly based: ACT/360, ACT/365, 30/360, 30/365. Let us call these rates $r_{ACT/360}$, $r_{ACT/365}$, $r_{30/360}$, and $r_{30/365}$.

$$\frac{\$100}{1+r_{ACT/360}\left(\frac{182}{360}\right)} = \$96.55,\ r_{ACT/360} = 0.0707$$

$$\frac{\$100}{1+r_{ACT/365}\left(\frac{182}{365}\right)} = \$96.55,\ r_{ACT/365} = 0.0717$$

$$\frac{\$100}{1+r_{30/360}\left(\frac{180}{360}\right)} = \$96.55,\ r_{30/360} = 0.0715$$

$$\frac{\$100}{1+r_{30/365}\left(\frac{180}{365}\right)} = \$96.55,\ r_{30/365} = 0.0725$$

Thus, the quoted rates would be different but the present values must be the same. Otherwise, one could engage in an arbitrage, buying the cheaper one and selling the more expensive one.