TEACHING NOTE 12-01:
PRICING AND VALUATION OF AMORTIZING INTEREST RATE SWAPS

Version date: August 1, 2013 C:\Users\Don\Documents\Work Folders\Class Material\Teaching Notes\TN12-01.docx

This Teaching Note on the pricing and valuation of amortizing interest rate swaps dovetails with TN97-06, which covers the pricing and valuation of (non-amortizing) interest rate and currency swaps. You might also wish to review TN05-01, The Pricing and Interest Sensitivity of Floating-Rate Securities, and TN13-01, Pricing and Valuation of Adjustable Interest Rate Swaps.

An amortizing swap is one in which the notional principal decreases through time. One type of amortizing swap is called an index amortizing swap, which amortizes according to a schedule specifying the rate of reduction of principal according to the level of the underlying rate. The lower the interest rate the greater the rate of amortization. This type of swap can serve somewhat to hedge certain amortizing instruments such as mortgages, which often are refinanced when interest rates are lower. The accelerated reduction of principal on the swap is designed to mimic the more rapid payoffs of mortgages when interest rates fall.[1]

Many companies take out amortizing floating-rate loans. To offer a hedge against interest rate increases, lenders often propose that borrowers simultaneously take out amortizing swaps that effectively convert these amortizing floating-rate loans into amortizing fixed-rate loans. The notion of an amortizing floating-rate loan can, however, be a source of some confusion. Typically the amortization on a loan such as a fixed-rate mortgage is endogenous. The fixed rate is the rate that sets the present value of the series of equivalent payments equal to the value of the loan. The amortization schedule is automatically created by subtracting the interest, calculated as the fixed rate applied to the loan balance, from the fixed payment. Thus, the amortization schedule is endogenous in that it is fully specified by the terms of the loan, i.e., the amount borrowed, the rate, and the number and frequency of payments.

[1]This note can also be used to price accreting swaps, which have appeared in the market. In this note, we specify loan balances at the scheduled payment dates throughout the life of the swap but make no assumptions that the balances are decreasing over time.

Variable rate mortgages often amortize by taking the first floating rate, calculating an amortization schedule for the entire loan at that rate, and then recalculating the amortization rate each time the floating rate is reset. Another way to amortize would be to set the amortization schedule based on the fixed rate and maintain it over the life of the loan. These methods are, however, only two of many possible ways in which an amortization schedule can be created for a variable-rate loan.

Many floating rate corporate loans have an arbitrary or completely exogenous schedule. In fact, the non-amortizing fixed-rate loan, in which 100% of the principal is repaid on the last payment date, is simply a special case of an arbitrary amortization schedule in which all of the amortization occurs on the last payment date. Other reasonable schedules are possible. A one-year corporate loan with quarterly payments could amortize at say 40% of the principal the first quarter, 30% the second quarter, 20% the third quarter, and 10% the last quarter. Clearly there are an infinite number of possible exogenous amortization schedules.

On a corporate loan, the lender decides on the amortization schedule and simply includes it in the loan documents. Thus, the schedule is fixed and known to all parties, though the factors that determine it might be known only to the lender. Many of these floating-rate loans will then be converted into fixed-rate loans with a swap, which must clearly be an amortizing swap. The net result is that an amortizing floating-rate loan plus an amortizing swap that offsets the floating payment on the loan must be equivalent to an amortizing fixed-rate loan.

Given that amortizing fixed rate-loans are fairly common and well-understood, this type of hedge might seem straightforward, but LIBOR discounting, which is often based on exact day counts, adds a complicating element. For example, an amortizing fixed-rate loan such as a mortgage with monthly payments almost always specifies that interest be computed as a monthly rate, which is referred to as 30/360 day count. That is, the mortgage rate is multiplied by 1/12 times the loan balance to determine the interest. LIBOR-based loans, however, almost never compute interest in this manner. Rather, the interest rate is multiplied by the actual number of days since the last payment and then divided by 360. Thus, in LIBOR-based loans, daily interest is usually paid for the exact number of days since the last interest payment was made. Hence, we cannot equate the standard amortizing fixed-rate loan to an amortizing fixed-rate loan based on LIBOR. Nonetheless, we can price a swap tied to an amortizing fixed-rate loan based on LIBOR.

Pricing the Amortizing Swap

Recall that we are interested in determining the fixed rate and the market value of a swap.[2] As we noted earlier, it is axiomatic to say that a borrower who has

an amortizing floating-rate loan

plus

an amortizing swap to pay fixed and receive floating

is equivalent to

an amortizing fixed-rate loan.

Because the floating rates offset, the fixed rate on the amortizing swap must be the fixed rate on the amortizing fixed-rate loan. Thus, if we find the fixed rate on the amortizing fixed-rate loan, we have determined the fixed rate on the amortizing swap. Let us denote that rate as r.[3]

From the swaps teaching note, 97-06, recall that we start at time 0 and have a series of payment dates denoted as times 1, 2, …, n. The number of days from time 0 to each respective date is d_1, d_2, d_n. The term structure of LIBOR of $r(0,d_i)$ implies a present value of $1 on each date d_i of $B(0,d_i) = 1/(1 + r(0,d_i)d_i/360)$ for all i = 1, …, n.[4] In general, for any two time points i and i + j, the interest rate is $r(d_i,d_{i+j})$, the number of days is $d_{i+j} – d_i$ and the discount factor is $B(d_i,d_{i+j}) = 1/(1 + r(d_i,d_{i+j})(d_{i+j} – d_i)/360)$.

Let us specify the exogenous amortization schedule on a loan of n payments in terms of the loan balances at times 0, 1, 2, …, n, as L(0), L(1), L(2), …, L(n) after the payment has been applied. Obviously L(0) is the original amount borrowed and L(n) = 0. Assuming a generic interest rate of r, the first loan payment is L(0) – L(1) (principal) and $L(0)rd_1/360$ (interest), reflecting the fact that there are d_1 days of interest accrued from the beginning of the loan. The second loan payment is L(1) – L(2) (principal) and

---

[2]As in TN97-06, we assume away any credit-related issues. These are important, but this note focuses strictly on interest rate risk.

[3]There is a similar point for non-amortizing swaps. We showed that the fixed rate on such a swap is the fixed rate on a par value bond with the same maturity and payment dates.

[4]Although we state that r is given and that $B(0,d_i)$ is derived, in reality it is the other way around. $B(0,d_i)$ is really the time value of money: what is $1 worth d_i days from now? The forces of a competitive market place make that determination. Then, given $B(0,d_i)$, r is implied. All interest rates in financial markets are determined in that manner, even though people tend to think the direction is reversed. Nonetheless, we will work with the implied interest rate as given and determine $B(0,d_i)$.

$L(1)r(d_2 - d_1)/360$ (interest), reflecting the fact that there are $d_2 - d_1$ days since the first payment. Proceeding in that manner, the last loan payment is L(n-1) – L(n) (principal) and $L(n\text{-}1)r(d_n - d_{n-1})/360$ (interest).

Thus, the i^{th} payment can be expressed as L(i-1) – L(i) (principal) and $L(i\text{-}1)r(d_i - d_{i-1})/360$ (interest), which can further be written as $L(i\text{-}1)(1 + r(d_i - d_{i-1})/360) - L(i)$. To find the value of all of the payments, each payments is discounted by the respective LIBOR discount factor $B(0,d_i)$. Thus, the present value of the payments is

$$\sum_{i=1}^{n}\left(L(i-1)\left(1+r\left(\frac{d_i-d_{i-1}}{360}\right)\right)-L(i)\right)B(0,d_i).$$

Of course this must equal the amount borrowed. Thus,

$$\sum_{i=1}^{n}\left(L(i-1)\left(1+r\left(\frac{d_i-d_{i-1}}{360}\right)\right)-L(i)\right)B(0,d_i)=L(0).$$

The fixed rate is, therefore, the solution:

$$r=\left(\frac{L(0)-\sum_{i=1}^{n}(L(i-1)-L(i))B(0,d_i)}{\sum_{i=1}^{n}L(i-1)(d_i-d_{i-1})B(0,d_i)}\right)360.$$

Three special cases are interesting. First, for equally spaced payments, $d_i - d_{i-1}$ = d for all i. Then the solution becomes

$$r=\left(\frac{L(0)-\sum_{i=1}^{n}(L(i-1)-L(i))B(0,d_i)}{\sum_{i=1}^{n}L(i-1)B(0,d_i)}\right)\left(\frac{360}{d}\right)$$
$$(\textit{for equally spaced payments}).$$

Second, in the non-amortizing interest rate swap teaching note, 97-06, we standardized the loan to a value of 1.0. Thus, with L(0) = 1, the formula becomes

$$r=\left(\frac{1.0-\sum_{i=1}^{n}(L(i-1)-L(i))B(0,d_i)}{\sum_{i=1}^{n}L(i-1)B(0,d_i)}\right)\left(\frac{360}{d}\right)$$
$$(\textit{for equally spaced payments and notional principal of } 1.0).$$

This formula look somewhat like the one developed for non-amortizing fixed-rate swaps. Whereas under those assumptions, the discounted factors are unweighted, exogenous amortization results in weighting the discount factors by their respective loan balances.

Third is when amortization is the standard case wherein no principal payments are made until the last interest payment in which the principal is paid in its entirety. Then, L(i) = L(0) for all i = 1, …, n-1 and L(n) = 0. Thus, in the numerator above, the expression L(i-1) – L(i) = L(0) – L(0) except in the last case, i = n, in which L(n-1) – L(n) = L(0) – 0 = L(0). For the case of a $1 loan, the formula then reduces to the formula for the standard case, which is given in TN97-06. Thus, we see that the formula derived in the present note is quite general.

See the appendix of this note for a numerical example using the same term structure used in the non-amortizing swap Teaching Note, 97-06.

Valuation of the Amortizing Swap

Valuation of the amortizing floating-rate swap can be done by determining the difference in the value of the amortizing floating-rate loan and the amortizing fixed-rate loan. The former minus the latter is the value of the amortizing floating-rate swap from the perspective of the floating receiver. Let us now derive the values of the amortizing floating- and fixed-rate loans.

As in the non-amortizing swaps Teaching Note, 97-06, we assume that the swap is initiated at time 0, and that we are now at time t, which is between time 0 and time 1. We have a new set of interest rates, $r(d_t,d_1)$, $r(d_t,d_2)$, …, $r(d_t,d_n)$ and a new set of discount bond prices, $B(d_t,d_1)$, $B(d_t,d_2)$, …, $B(d_t,d_n)$. The fixed rate has already been set and is known to be r. The most recent floating rate is $r(0,d_1)$ and represents the rate at which the next floating payment will be made.

For a floating-rate loan with no amortization, we invoked the attractive feature that the value of such a loan always returns to its par value on its rate reset dates. For an amortizing floating-rate loan, this condition is not true but an analogous and considerably more general result obtains: its value will simply be the principal balance of the loan. Let us prove that result.

We start at time n-1, the next-to-last payment date. The upcoming and final payment at n will be L(n-1) – L(n) (principal) and $L(n-1)r(d_{n-1},d_n)(d_n - d_{n-1})/360$

(interest), which can be expressed as L(n-1)(1 + r(d_{n-1},d_n)(d_n – d_{n-1})/360)) – L(n). The present value of this amount at n-1 is

$$(L(n-1)(1+r(d_{n-1},d_n)(d_n-d_{n-1})/360)-L(n))B(d_{n-1},d_n).$$

This simplifies to

$$L(n-1)-L(n)B(d_{n-1},d_n).$$

In this special case, we have one further simplification, that L(n) = 0, so this reduces to L(n-1).

Now we back up to n-2. The upcoming payment at n-1 will be L(n-2) – L(n-1) (principal) and L(n-2)r(d_{n-2},d_{n-1})(d_{n-1} – d_{n-2})/360 (interest), which can be expressed as L(n-2)(1 + r(d_{n-2},d_{n-1})(d_{n-1} – d_{n-2})/360) – L(n-1). We also have a claim on the value at n-1, which we showed above is L(n-1). The sum of these two expressions is

$$\begin{aligned}&L(n-2)(1+r(d_{n-2},d_{n-1})(d_{n-1}-d_{n-2})/360)-L(n-1)+L(n-1)\\&=L(n-2)(1+r(d_{n-2},d_{n-1})(d_{n-1}-d_{n-2})/360).\end{aligned}$$

Then we discount this term from n-1 to n-2 to obtain

$$(L(n-2)(1+r(d_{n-2},d_{n-1})(d_{n-1}-d_{n-2})/360))B(d_{n-2},d_{n-1})=L(n-2)$$

If we continue working backwards, we obtain that the value of the amortizing floating-rate loan at any time i is L(i). This should make sense. It is the amount of principal to be repaid. Thus, this result is far more general than the result we used in TN97-06, that the value of a floating-rate loan returns to par value at each payment date. In general, the value of a loan returns to the loan balance on each payment date. Of course, it must because the loan balance is by definition the value of the loan. For non-amortizing floating-rate loans, par value *is* the loan balance until the final payment.

In the swaps teaching note, we positioned ourselves at time t, which is between time 0 and time 1. The next payment will occur at time 1 and will consist of interest of L(0)r(0,d_1)d_1/360 and principal of L(0) – L(1), which can be written as L(0)(1 + r(0,d_1)d_1/360) – L(1) – L(1) = L(0)(1 + r(0,d_1)d_1/360). Thus, the value of the floating rate loan at time t will be the present value of this amount discounted from d_1 back to t:

$$V_{AFLRL}(t)=(L(0)(1+r(0,d_1)d_1/360))B(d_t,d_1).$$

This formula is set up at time t, which occurs after the loan initiation date and before the first interest payment. It can be generalized by setting time 0 above to whatever is the previous interest payment date. Note that $d_1/360$ in the numerator is the number of days in the full payment period. Thus, it would be $(d_i - d_{i-1})/360$ for valuation at time t between i and i-1. Note also that if we rolled t back to 0, the formula would become L(0), as it should. Hence, we know that this formula gives the correct value of the amortizing floating-rate loan at any time from 0 to n.

Now we value the amortizing fixed-rate loan at time t. Its formula is quite simple:

$$V_{AFXRL}(t) = \sum_{i=1}^{n}\left(L(i-1)\left(1 + r\left(\frac{d_i - d_{i-1}}{360}\right)\right) - L(i)\right) B(d_t, d_i).$$

This is the same formula we used above in solving for the fixed rate, except that we are now at time t instead of time 0 and we do not force its value to equal anything in particular.

Thus, from the perspective of the floating-rate receiver, the value of the amortizing floating-rate swap at t is

$$\begin{aligned} V_{ASWAP}(t) &= V_{AFLRL} - V_{AFXRL} \\ &= \left(L(0)(1 + r(0,d_1)d_1/360)B(d_t,d_1)\right) \\ &\quad - \left(\sum_{i=1}^{n}\left(L(i-1)\left(1 + r\left(\frac{d_i - d_{i-1}}{360}\right)\right) - L(i)\right) B(d_t, d_i)\right). \end{aligned}$$

Appendix

Let us now do some numerical calculations.[5] Using the same example in the non-amortizing swaps Teaching Note, 97-06, we have the term structure and discount factors at time 0 as[6]

r(0,1) = 0.08, r(0,2) = 0.09, r(0,3) = 0.10

B(0,1) = 1/[1 + 0.08(360/360)] = 0.9259

B(0,2) = 1/[1 + 0.09(720/360)] = 0.8475

B(0,3) = 1/[1 + 0.10(1080/360)] = 0.7692.

Thus, $d_1 = 360$, $d_2 = 720$, and $d_3 = 1080$. Let the loan be for \$100,000 and assume that the amortization is 40% of principal at time 1, 35% and time 2, and 25% at time 3. Thus, L(0) = \$100,000, L(1) = \$60,000, L(2) = \$25,000, and L(3) = \$0.[7]

---

[5]All calculations have been done a spreadsheet, so they may differ slightly from calculations done by hand.
[6]This is the U. S. term structure in TN97-06.
[7]It is apparent that the loan balances could be standardized to a \$1.0 loan ($L_0 = 1$), as in TN97-06, and the balances be expressed as percentages. Thus, L(1) would be 0.6, L(2) would be 0.25, and L(0) would be

Recall that the formula for the fixed rate on the swap is

$$r = \left(\frac{L(0) - \sum_{i=1}^{n} (L(i-1) - L(i)) B(0, d_i)}{\sum_{i=1}^{n} L(i-1)(d_i - d_{i-1}) B(0, d_i)} \right) 360.$$

The summation in the numerator is

$$\begin{aligned} &\sum_{i=1}^{n} (L(i-1) - L(i)) B(0, d_i) \\ &= (\$100,000 - \$60,000)0.9259 \\ &\ +(\$60,000 - \$25,000)0.8475 \\ &\ +(\$25,000 - \$0)0.7692 \\ &= \$85,929 \end{aligned}$$

The summation in the denominator is

$$\begin{aligned} &\sum_{i=1}^{n} L(i-1)(d_i - d_{i-1}) B(0, d_i) \\ &= \$100,000(360 - 0)0.9259 \\ &\ +\$60,000(720 - 360)0.8475 \\ &\ +\$25,000(1080 - 720)0.7692 \\ &= \$58,561,495 \end{aligned}$$

Therefore, the rate is

$$r = \left(\frac{\$100,000 - \$85,929}{\$58,561,495} \right) 360 = 0.0865$$

Note that for the non-amortizing swap, the rate was higher at 0.0908.

To value the swap, let us move forward 180 days, at which time we confront a new term structure of 180, 540, and 720 days, from TN97-06:

r(0.5,1) = 0.082, r(0.5, 2) = 0.094, r(0.5,3) = 0.105

B(0.5,1) = 1/[1 + 0.082(180/360)] = 0.9606

B(0.5,2) = 1/[1 + 0.094(540/360)] = 0.8764

B(0.5,3) = 1/[1 + 0.105(900/360)] = 0.7921

First let us find the value of the amortizing floating-rate note. The upcoming payment will be at the original 360-day rate of 0.08. The value of the loan is

---

0.0. As with the case of non-amortizing floating-rate swaps, the loan balance is not relevant in determining the fixed rate.

$$\begin{aligned} V_{AFLRL}(t) &= (L(0)(1 + r(0, d_1)d_1 / 360))B(d_t, d_1) \\ &= (\$100,000(1 + .08(360/360))0.9606 \\ &= \$103,746 \end{aligned}$$

The value of the amortizing fixed-rate loan is

$$\begin{aligned} V_{AFXRL}(t) &= \sum_{i=1}^{n} \left(L(i-1)\left(1 + r\left(\frac{d_i - d_{i-1}}{360}\right)\right) - L(i) \right) B(d_t, d_i) \\ &= (\$100,000(1 + 0.0865(360 - 0)/360) - \$60,000)0.9606 \\ &\quad + (\$60,000(1 + 0.0865(720 - 360)/360) - \$25,000)0.8764 \\ &\quad + (\$25,000(1 + 0.0865(1080 - 720)/360) - \$0)0.7921 \\ &= \$103,472 \end{aligned}$$

Thus, the value of the amortizing floating-rate swap is

$$V_{ASWAP}(t) = \$103,746 - \$103,472 = \$274.$$

This is 0.27% of notional. Note that the value for the standard swap was 0.68% of notional. Of course, these values need not be the same and either can be higher, depending on the rate of amortization.

These formulas would give the same results as in TN97-06 for the non-amortizing loan by assuming the loan balance is the full amount of the loan until the last payment, such that L_3 = \$0.