TN07-01:
THE BERNOULLI PARADOX

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The Bernoulli paradox, also sometimes called the St. Petersburg Paradox, is one of the most famous puzzles of economics. It is oftentimes used to show why expected value is a poor indicator of how much one would pay to engage in a risky investment. The original article was written by the Swiss mathematician Daniel Bernoulli.[1] In this paper, cited in the references below, he noted that his mathematician brother Nicolas Bernoulli had posed an interesting problem to the French Mathematician Pierre Rémond Monmort. Bernoulli's paper was presented at a gathering of mathematicians in St. Petersburg, Russia in 1738. Originally written in Latin, it has been translated into English by Professor Louise Sommer of American University and later published in the prestigious economics journal, *Econometrica*. See the references for details.

The problem is posed as follows:

> *Peter tosses a coin and continues to do so until it should land on "heads" when it comes to the ground. He agrees to give Paul one ducat if he gets "heads" on the very first throw, two ducats if he gets it on the second, four if on the third, eight if on the fourth, and so on, so that with each additional throw the number of ducats he must pay is doubled.*[2]
>
> (See *Econometrica* version, p. 31)

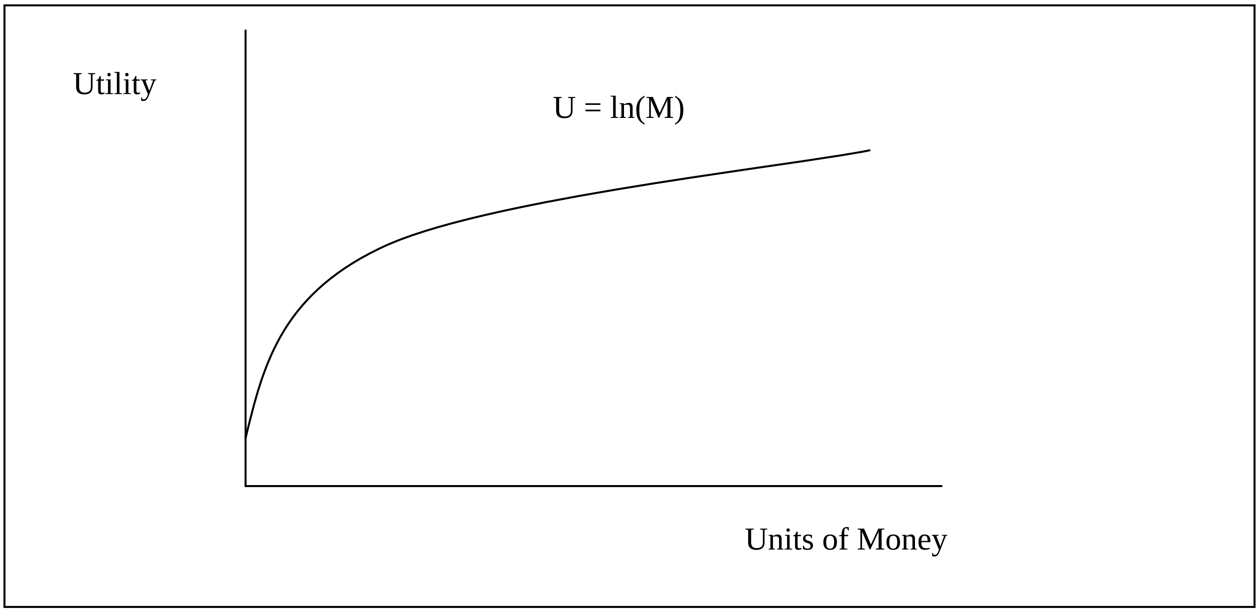
The question is how much Paul would play to pay the game? A simple calculation reveals that the expected payoff is infinite:

$$E(V) = \left(\frac{1}{2}\right)2^0 + \left(\frac{1}{2}\right)^2 2^1 + \left(\frac{1}{2}\right)^3 2^2 + \ldots = \infty$$

Thus, Paul should be willing to pay an infinite amount of money to play this game? But clearly, Paul and indeed no one would pay an infinite amount. Bernoulli proposed a solution based on the notion of expected utility.

[1]Bernoulli was technically born in the Netherlands but lived most of his life in Switzerland.
[2]A ducat was a unit of currency widely used in Europe at one time.

Paul, he reasoned, receives utility from money. With the money he can buy what he needs and wants. But each additional unit of money brings less and less satisfaction. Bernoulli proposed that utility is a logarithmic function of the amount of money, as shown below. Thus, more money will always lead to more utility but the marginal value of an additional unit of money is less than that of the previous unit of money. This relationship is illustrated below. Note that utility always increases with money but does so at a decreasing rate.



Using log utility, we can resolve the St. Petersburg Paradox. We first find the expected utility of playing the game. Expected utility is found by taking each outcome, determining the utility of that outcome, and multiplying by the probability of that outcome. Then we add up the results:

$$E(U) = \frac{1}{2}\ln(1) + \frac{1}{4}\ln(2) + \frac{1}{8}\ln(4) + \ldots$$

This looks like a hard problem but can actually be easily solved. Let us write it out in general terms.

$$E(U) = \left(\frac{1}{2}\right)^1 \ln(2^0) + \left(\frac{1}{2}\right)^2 \ln(2^1) + \left(\frac{1}{2}\right)^3 \ln(2^2) + \ldots$$

This can be re-written as

$$E(U) = \sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^{i+1}\ln(2^i) = \sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^{i+1} i\ln 2$$
$$= \ln 2\left(\frac{1}{2}\right)\sum_{i=0}^{\infty} i\left(\frac{1}{2}\right)^{i}$$

The summation is an infinite series with a known solution:

$$\sum_{i=1}^{\infty} ix^i = \frac{x}{(1-x)^2}$$

Thus, we have

$$E(U) = \ln 2\left(\frac{1}{2}\right)\frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = \ln 2 = 0.6938.$$

Thus, the expected utility is 0.6938. This means that the gamble has expected utility of 0.6938.

The amount one should pay is the amount of money with the same expected utility. How much money has the same expected utility as the gamble? This is easily found:

$$\ln(M) = 0.6938$$
$$M = e^{0.6938} \simeq 2.00$$

That is, if a person parted with two units of the currency, he would be giving up the same expected utility as taking the gamble. Hence, the person should be willing to pay two units of currency to play.

References

Bernoulli, Daniel. "Specimen Theoriae Novae de Mensura Sortis." *Commentarii Academiae Scientiarium Imperialis Petropolitanae*, Tomus V [*Papers of the Imperial Academy of Sciences in Petersburg*, Vol. V], 1738, pp. 175-192.

Bernoulli, Daniel. "Exposition of a New Theory on the Measurement of Risk." *Econometrica*, 22 (January, 1954), 23-36. Translated by Dr. Louise Sommer of American University.

The St. Petersburg paradox has been written about extensively in the literature on utility theory in the 20th century.