MBA Teaching Note 18-02
Exponentials, Logarithms, and Continuous Compounding in Finance

Exponentials and logarithms are commonly used in finance, often for reasons that are simply not well understood. In this note, we explain exponentials and logarithms, and then we should how they are often used in finance.

Let us start with logarithms. A logarithm is a number that represents an exponent to which a base number is raised to obtain he given number. For example, if we say that the log of *x* is *y*, it means that some number, the base, raised to the power *y* equals *x*. Notice that we did not say what the base number is. There are two common base numbers, 10 and *e*. Ten is commonly used because it is the base of the decimal number system, but *e* is often more convenient and has a special context in finance.

Common Logarithms

Logarithms that use the base 10 are referred to as common logarithms or just common logs. They are typically denoted in the form $\log_{10}$. Thus, we might wish to know the common log of a number, say 100. Recall that we defined a logarithm as the exponent to which the base is raised to equal the number. In the case of the common log of a 100, we want to know the power to which 10 must be raised to equal 100. Obviously, that is 2. Hence, $\log_{10}100 = 2$. Any positive number can have a common log, so our common logs do not have to be integers. For example, consider the $\log_{10}150$. Here, we are expressing the power to which 10 must be raised to equal 150.

Typically to get such an answer, we have our calculator do the work.[1] It would tell us that the answer is 2.1761, which must mean that

$$10^{2.1761} = 100$$

But why do we know this is true? In order to understand why this is true, we have to recall what it means to raise a number to a fractional power. To see how this works, let us simply recall a couple of fundamental rules about exponents that are typically learned in high school:

$$\sqrt[k]{a} = a^{\frac{1}{k}}$$

$$a^{-k} = \frac{1}{a^k}$$

As an example, if we want the square root (2nd root) of 10, we raise 10 to the power ½.

$$\sqrt[2]{10} = 10^{\frac{1}{2}} = 3.1623$$

So, if we want to take the k^{th} root of something, we raise it to the power $1/k$.

Another important rule is

$$a^{x+y} = a^x a^y$$

$$a^{x-y} = a^x a^{-y}$$

Applying that rule, let work a slightly harder problem, such as the one we started above, raising 10 to the power 2.1761. First, note that 2.1761 = 2 + 0.1761. Hence, in our problem, we have

$$10^2 10^{0.1761} = 10^{2.1761}$$

Hence, we can first find 10^2, which we obviously know is 100. Then we need $10^{0.1761}$, which we do not know. We can, however, write it as follows:

$$10^{\frac{1,761}{10,000}}$$

[1]In the pre-calculator days, tables of logarithms were always available in the backs of math books.

And from the above rule, we can write this as

$$10^{\frac{1,761}{10,000}} = 10^{1,761} 10^{-10,000}$$

First, let us do $10^{-10,000}$. Hence,

$$10^{-10,000} = \frac{1}{10^{10,000}} = 1.0002$$

Then we raise this number to the power of 1,761

$$1.0002^{1,761} = 1.5$$

Of course, machines now do this for us. We can simply enter 10 and raise it to the power 0.1761, but what we have seen here is why this works.

Natural Logarithms

Natural logarithms are sometimes called natural logs or Naperian logs, named after the mathematician John Napier. They use the base *e*, which is a special number called a mathematical constant that has the approximate value of 2.7181828459 when carried out to ten decimal places.[2] We often shorten it to 2.718. This number is widely found in the sciences but also has a special place in finance, which we will show later. In fact, the number was said to have been discovered when Swiss mathematician, Jacob Bernoulli, was attempting to understand compound interest. This value *e* is also sometimes called Euler's number, after the Swiss mathematician Leonhard Euler. Let us look at how the number is defined and how it is related to finance.

One definition of *e* has it as the sum of an infinite series

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{2*1} + \frac{1}{3*2*1} + \ldots$$

This formula, of course, is an infinite series, so it is not possible to add up all of these numbers. The table below shows the value calculated to ten decimal places for $n = 10$.

n	Value	Sum
0	1.0000000000	1.0000000000
1	1.0000000000	2.0000000000
2	0.5000000000	2.5000000000
3	0.1666666667	2.6666666667
4	0.0416666667	2.7083333333
5	0.0083333333	2.7166666667
6	0.0013888889	2.7180555556
7	0.0001984127	2.7182539683
8	0.0000248016	2.7182787698
9	0.0000027557	2.7182815256
10	0.0000002756	2.7182818011

Note how the number is stabilizing. The formal definition of *e* is

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

And the infinite series is well-known to approximate this function.

Now, let us consider the function e^x, sometimes written as $\exp(x)$. It is formally equal to

[2]The decimal part of the number, called the mantissa, does go on forever.

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

Using the factorial as we did above, this is approximated as

$$\begin{aligned} e^x &= \lim_{n\to\infty}\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots+\frac{1}{n!}x^n\right) \\ &= \sum_{i=0}^{\infty} x^i/i! \end{aligned}$$

As an example, suppose we want e^3. We know from using our calculator that e^3 is 20.0855. Using the factorial above, we have

i	$x^i/i!$	Sum
0	1.0000	1.0000
1	3.0000	4.0000
2	4.5000	8.5000
3	4.5000	13.0000
4	3.3750	16.3750
5	2.0250	18.4000
6	1.0125	19.4125
7	0.4339	19.8464
8	0.1627	20.0092
9	0.0542	20.0634
10	0.0163	20.0797
11	0.0044	20.0841
12	0.0011	20.0852
13	0.0003	20.0855
14	0.0001	20.0855
15	0.0000	20.0855
16	0.0000	20.0855
17	0.0000	20.0855
18	0.0000	20.0855
19	0.0000	20.0855
20	0.0000	20.0855

And we see that the sum is stabilizing at 20.0855.

The Exponential Function and Compound Interest

Now let us take a look at how interest is calculated. Suppose we borrow \$1 for one year at the annual rate *r*. After one year, we would pay back

$$\$1(1+r) = 1+r$$

Now, suppose the rate r was stated as having semiannual compounding, such that after one year we would have

$$\$1\left(1+\frac{r}{2}\right)\left(1+\frac{r}{2}\right) = \$1\left(1+\frac{r}{2}\right)^2$$

Suppose the annual rate is compounded quarterly. Then after one year we would have

$$\$1\left(1+\frac{r}{4}\right)\left(1+\frac{r}{4}\right)\left(1+\frac{r}{4}\right)\left(1+\frac{r}{4}\right)=\$1\left(1+\frac{r}{4}\right)^{4}$$

Or monthly,

$$\$1\left(1+\frac{r}{12}\right)\left(1+\frac{r}{12}\right)\left(1+\frac{r}{12}\right)\left(1+\frac{r}{12}\right)........=\$1\left(1+\frac{r}{12}\right)^{12}$$

And how about daily,

$$\$1\left(1+\frac{r}{365}\right)\left(1+\frac{r}{365}\right)\left(1+\frac{r}{365}\right)\left(1+\frac{r}{365}\right)........=\$1\left(1+\frac{r}{365}\right)^{365}$$

It should be apparent that given the same annual rate, the more frequent the compounding, the more money you would have after a given period of time as shown below for the case of $1 at an annual rate of 6%.

Compounding periods per year	Money after one year
1	1.0600000000
2	1.0609000000
4	1.0613635506
12	1.0616778119
365	1.0618313107

In general, let p = the number of compounding periods per year. Then the amount of money in the account after one year per $1 deposited is

$$\left(1+\frac{r}{p}\right)^{p}$$

We might want to generalize this problem to a certain number of years, say y. Then

$$\left(1+\frac{r}{p}\right)^{py}$$

Of course, there is clearly a periodic (monthly, quarterly, daily) rate that provides the same future value as the annual rate. For example, consider the case of p compounding periods and y years. We noted in the equation above the amount of money we would have. To express this annual rate as its semiannual equivalent, we would say the semiannual rate is $r/p = r_{sa}$. The equivalent annual rate for y years would be

$$\left(\left(1+r_p\right)^{yp}\right)^{\frac{1}{y}}-1=r_{eap}$$

Note, however, that the exponent yp and its exponent $1/y$ result in a cancellation, leaving the above as

$$(1+r_p)^{p}-1=r_{eap}$$

And thus, r_{eap} is the equivalent annual rate for the periodic rate r. An example is shown below for $1 invested at 12% with various compounding periods.

Compounding periods per year	Future value	Periodic rate	Equivalent annual rate
1	1.1236000000	0.0600000000	0.0600000000
2	1.1255088100	0.0300000000	0.0609000000
4	1.1264925866	0.0150000000	0.0613635506
12	1.1271597762	0.0050000000	0.0616778119
365	1.1274857323	0.0001643836	0.0618313107

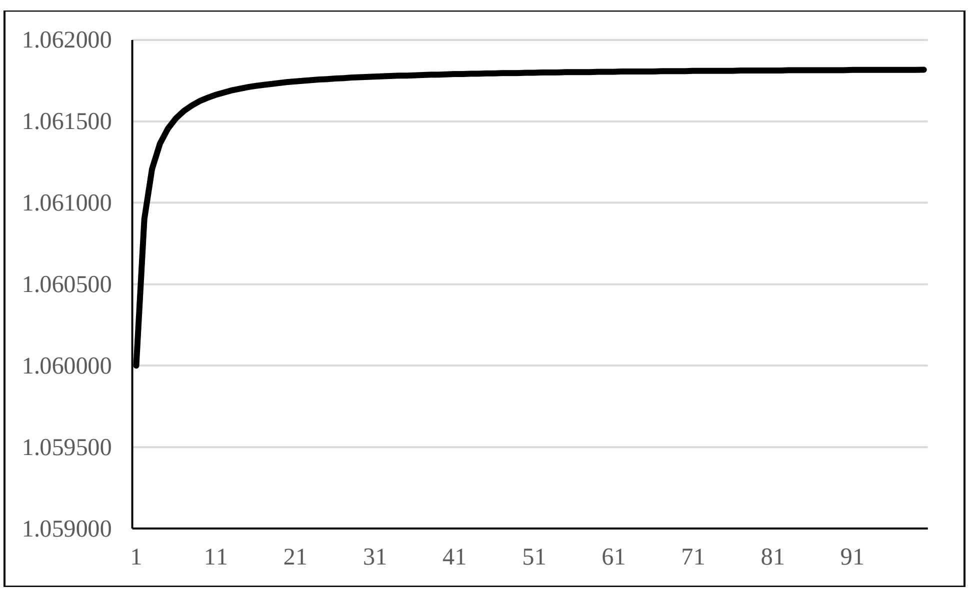
Let us restate the relationship between \$1 compounded for one year at an annual rate *r* for *p* and the equivalent annual rate as

$$\left(1+\frac{r}{p}\right)^{p}=1+r_{eap}$$

Notice to now that the compounding frequency has been no faster than daily, but there is no reason why we cannot specify an even more frequent compounding period. Suppose we specify the number of compound periods as *n*, an unspecified amount and let *n* grow increasingly large. Then as *n* approaches infinity, we say that we are compounding instantaneously. Accordingly, we call this continuous compounding. The amount of money we shall have after one year having compounded at the instantaneous rate of *r*/*n* for *n* periods is

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n}$$

This looks very similar to the definition we had of *e* we gave above, except that we had 1 instead of *r*. The graph below shows values of this expression for *n* up to 100. It stabilizes at around 1.0618. As it turns out, 1.0618 is approximately, exp(0.06).



Using our previous example about how much money we would have compounded at various intervals, 6% compounded for two years continuously would produce \$1.1274968516.

With continuous compounding, this is exp(0.06(2))= $1.1274968516. Continuous interest is little more interest but not much.

Continuously Compounded Rates of Return

If you purchased a stock at a price of $100 and one year later, the price was $110, what is your rate of return? Your instincts may say 10%. After all, the price went up 10%. But it should be apparent, that the choice of a compounding period could result in the return being expressed as a different rate, depending on the frequency of compounding. For example, consider the relationship below, which expresses the rate that it would take to raise 100 to 110 in one year with semiannual compounding,

$$100\left(1+\frac{r_p}{2}\right)^2 = 110$$

Note that we are implying that the periodic rate of r_p is a semiannual rate, as suggested by the fact that we divided the rate in half and we compound twice a year. From the above, the rate would be

$$r_p = 2\left(\sqrt{\frac{110}{100}} - 1\right) = 0.0976$$

Thus, one could just as easily say that a stock that went from $100 to $110 in one year grew at a rate of 9.76% compounded quarterly.

We could also express the rate of return as a quarterly, monthly, or daily rate. Now, let us carry this further to use continuous compounding. Here the math is simple. The continuous rate, r_c, is

$$r_c = \ln(1+r)$$

And in this case,

$$r_c = \ln(1.10) = 0.0953$$

Thus, if $100 compounds continuously for one year and grows to $110, the continuously compounded rate is 9.53%. This means that $100 grew to $110 at a rate of 9.53% when compounding continuously.

Continuous compounding is often used to express investment returns. One reason is that it has been shown that continuously compounded returns are closer to being normally distributed. In any case, here are the continuously compounded analogs of returns from -10% to +10% in half percent increments.

r	r_c	diff
-0.1000	-0.1054	0.0054
-0.0950	-0.0998	0.0048
-0.0900	-0.0943	0.0043
-0.0850	-0.0888	0.0038
-0.0800	-0.0834	0.0034
-0.0750	-0.0780	0.0030
-0.0700	-0.0726	0.0026
-0.0650	-0.0672	0.0022
-0.0600	-0.0619	0.0019
-0.0550	-0.0566	0.0016
-0.0500	-0.0513	0.0013
-0.0450	-0.0460	0.0010
-0.0400	-0.0408	0.0008
-0.0350	-0.0356	0.0006
-0.0300	-0.0305	0.0005
-0.0250	-0.0253	0.0003
-0.0200	-0.0202	0.0002
-0.0150	-0.0151	0.0001
-0.0100	-0.0101	0.0001
-0.0050	-0.0050	0.0000
0.0000	0.0000	0.0000
0.0050	0.0050	0.0000
0.0100	0.0100	0.0000
0.0150	0.0149	0.0001
0.0200	0.0198	0.0002
0.0250	0.0247	0.0003
0.0300	0.0296	0.0004
0.0350	0.0344	0.0006
0.0400	0.0392	0.0008
0.0450	0.0440	0.0010
0.0500	0.0488	0.0012
0.0550	0.0535	0.0015
0.0600	0.0583	0.0017
0.0650	0.0630	0.0020
0.0700	0.0677	0.0023
0.0750	0.0723	0.0027
0.0800	0.0770	0.0030
0.0850	0.0816	0.0034
0.0900	0.0862	0.0038
0.0950	0.0908	0.0042
0.1000	0.0953	0.0047

Finance students must be comfortable with continuous compounding, exponentials, and logarithms, as they facilitate the understanding of interest and investment performance.