MBA Teaching Note 14-01
Growth, Retention, and Reinvestment Rates in the Discounted Cash Flow Model

The well-known discounted cash flow model of equity valuation is the equation

$$P_i = \frac{D_{i+1}}{k-g}$$

where P_i is the current stock price, D_{i+1} is the next period dividend, k is the discount rate or required rate of return and g is the growth rate. Of course, we often position ourselves at time zero so that $i = 0$ and $i + 1 = 1$, giving us the current stock price. In general, the price is always positioned at one point in time and the dividend is one period later. This model is known to result from the assumption that earnings per share grow at the constant rate g forever and that the firm pays a constant percentage of its earnings as dividends. These assumptions also imply that dividends and the price of the stock grow at that same rate, g.

When the company retains a percentage of its earnings and reinvests those earnings, it earns a certain rate of return. With the assumption of a constant retention rate and a constant rate of return on those reinvested earnings, we can express the growth rate, g, as a function of the retention rate and reinvestment return. This note fills in how this operation is done. It then explains how growth can be valuable or harmful to shareholders.

Why the Growth Rate is the Retention Rate times the Reinvestment Rate

Let b represent the retention rate. Thus, by definition $D_i = (1 - b)E_i$. That is, the dividend is the percentage remaining, $1 - b$, of the earnings after the complementary percentage, b, is retained and reinvested. The expression $1 - b$ is called the payout rate or ratio. Let r represent the rate of return earned on reinvested earnings, which we will shorten and call the reinvestment rate. We will now prove that the growth rate, g, is the product of the retention rate, b, and the reinvestment rate, r.

First, as we noted, g, the growth rate is defined as the growth rate in earnings. Thus, by definition for any level of earnings, E_i, the growth rate is the ratio of the change in earnings divided by the earnings,

$$g = \frac{\Delta E_i}{E_i}.$$

Let us specify that a company undertakes new investment in period i of ΔI_i. By definition, the change in earnings from one period to the next is the new investment times the reinvestment rate,

$$\Delta E_i = \Delta I_i r.$$

And by definition, the new investment, ΔI_i, is the current earnings times the retention rate,

$$\Delta I_i = E_i b.$$

The above two equations combine to imply that

$$\frac{\Delta E_i}{E_i} = br.$$

Given that the growth rate is the change in earnings divided by the earnings, we have now proven that

$$g = br.$$

Let us now show that dividends will also grow at the rate, g. The change in dividends is the change in earnings times the payout rate,

$$\Delta D_i = \Delta E_i (1-b).$$

Divide through by the current dividend, D_i:

$$\frac{\Delta D_i}{D_i} = \frac{\Delta E_i}{D_i}(1-b)$$

On the right-hand side, we can substitute $E_i(1 - b)$ for the D_i in the denominator.

$$\frac{\Delta D_i}{D_i} = \frac{\Delta E_i}{E_i(1-b)}(1-b) = \frac{\Delta E_i}{E_i}$$

By definition, the right-hand side is now equal to g. Thus,

$$\frac{\Delta D_i}{D_i} = \frac{\Delta E_i}{E_i} = g.$$

We have now shown that dividends also grow at the rate g. This fact is obvious, however, if you simply consider that earnings grow at the rate g and dividends are a constant percentage of earnings. Thus, dividends must also grow at the rate g.

We now show that the stock price will also grow at the rate g. Working with the stock price equation, we can express the sequence of stock prices as

$$P_i = \frac{D_{i+1}}{k-g},\ P_{i+1} = \frac{D_{i+2}}{k-g},\ \ldots$$

Express the ratio of the stock prices, $P_{i+1}/P_i - 1$. Thus,

$$\frac{P_{i+1}}{P_i} = \frac{D_{i+2}/(k-g)}{D_{i+1}/(k-g)} = \frac{D_{i+2}}{D_{i+1}} = \frac{D_{i+1} + \Delta D_i}{D_{i+1}} = \frac{\Delta D_i}{D_{i+1}} + 1 = \mathrm{g} + 1.$$

The left-hand side is the growth rate in price plus 1.

$$\frac{P_{i+1}}{P_i} = \frac{P_i + \Delta P_i}{P_i} = \frac{\Delta P_i}{P_i} + 1 = \mathrm{g} + 1.$$

Thus, the price also grows at the rate g. This, too, is apparent upon remembering that the price is the sum of the present values of all of the expected future dividends. If those dividends are growing at a constant rate, so too must the price.

Good Growth and Bad Growth

As we showed, the earnings, dividends, and stock price grow at the rate g. Nonetheless, that grow is not necessarily good growth. Suppose, for example, that companies earned only the risk-free rate on their reinvested earnings. It is obvious that their earnings would grow but clearly such growth would not be adequate to satisfy stockholders who are accepting risk by investing in the company and require a higher return than the risk-free rate.

Brealey, Myers, and Allen, authors of a well-known corporate finance textbook (*Corporate Finance*, McGraw-Hill publishers), express this concept by decomposing the stock price into the present value of a constant level of earnings and the present value of the growth, *PVGO*, called *present value of growth opportunities*. I call it the value of reinvested earnings (VRE). It is derived as follows:

$$\begin{aligned} P_i &= \frac{D_{i+1}}{k-g} \\ &= VRE + \frac{E_{i+1}}{k}. \end{aligned}$$

The second term on the right-hand side, E_{i+1}/k, is the component of the stock price that arises from the assumption that earnings do not grow. It is clearly the present value of a perpetual stream of constant earnings of E_{i+1}. The term VRE is, thus, the present value of the growth opportunities. Let us see what that term consists of:

$$\begin{aligned} VRE &= P_i - \frac{E_{i+1}}{k} \\ &= \frac{D_{i+1}}{k-g} - \frac{E_{i+1}}{k}. \end{aligned}$$

If we substitute, $E_{i+1}(1 - b)$ for D_{i+1}, we can now derive an expression for VRE.

$$VRE = \frac{D_{i+1}}{k-g} - \frac{E_{i+1}}{k}$$
$$= \frac{E_{i+1}(1-b)}{k-g} - \frac{E_{i+1}}{k}.$$

Now substitute *br* for *g*.

$$VRE = \frac{E_{i+1}(1-b)}{k-br} - \frac{E_{i+1}}{k}.$$

This result implies that

$$VRE > 0 \ if \ \ E_{i+1}(1-b)k > (k-br)E_{i+1}.$$

A little algebra reveals that

$$E_{i+1}(1-b)k > (k-br)E_{i+1}$$
$$if$$
$$r > k.$$

Thus, *VRE* is positive if the firm reinvests its earnings at a rate, *r*, greater than the required rate of the shareholders, *k*. This result should make sense: if the firm cannot earn at least the required rate of return of the shareholders when it reinvests its earnings, it is wasting the shareholders' money reinvesting those earnings. It should pay the earnings out in the form of a higher dividend. In fact, it can be shown that if $r < k$, the stock price will increase if the retention rate, *b*, is reduced, meaning that the dividend is increased. This statement is equivalent to the statement that if *b* is reduced, *VRE* will increase. This result can be demonstrated by differentiating *VRE* by *b*:

$$\frac{\partial VRE}{\partial b} = \frac{\partial}{\partial b}\left(\frac{E_{i+1}(1-b)}{k-br} - \frac{E_{i+1}}{k}\right) = \frac{\partial}{\partial b}\left(\frac{E_{i+1}(1-b)}{k-br}\right)$$
$$= \frac{(k-br)(-E_{i+1}) - E_{i+1}(1-b)(-r)}{(k-br)^2}$$
$$= \frac{-E_{i+1}k + E_{i+1}br + E_{i+1}r - E_{i+1}br}{(k-br)^2}$$
$$= \frac{-E_{i+1}k + E_{i+1}r}{(k-br)^2} = -E_{i+1}\frac{k-r}{(k-br)^2}.$$

The expression is obviously negative if $k > r$ and positive if $r > k$. Thus, if the firm cannot earn the shareholders' required rate ($r < k$), an increase in the retention rate will reduce the price, so a decrease in the retention rate, meaning an increase in the payout rate, will increase the price.