MBA Teaching Note 07-05
Geometric and Arithmetic Rates of Return

One of the most important tools one acquires when studying finance is the ability to analyze investment performance information. One of the first contexts in which we encounter such data is in studying the historical performance of the stock market. In order to correctly interpret this data, we need to understand the concept of the arithmetic and geometric mean returns. But first we must understand the notion of a rate of return.

Rates of Return

Consider a sequence of N stock prices:

$$S_0, S_1, S_2, ..., S_N$$

This series represents observed prices of a stock and are equally spaced on a daily, weekly, monthly, quarterly, or yearly basis. We want to know how this stock performed. First we need the rates of return.

The rate of return is the percentage change in price. For any time point t, the rate of return at time t is expressed as[1]

$$r_t = \frac{S_t - S_{t-1}}{S_{t-1}}.$$

There is nothing wrong with doing it this way, and indeed it is quite intuitive. Most people understand the concept of a percentage change. But the above formula requires more keystrokes when entering it into a calculator. A short-cut is

$$r_t = \frac{S_t}{S_{t-1}} - 1.$$

Here the original price is entered only once. We call the ratio S_t/S_{t-1} the price relative.[2] In the sequence of prices above, the first return is

$$r_1 = \frac{S_1}{S_0} - 1.$$

Returning to our sequence of prices, the first return, r_1, is followed by the second return, r_2, which is followed by the third return, r_3. This process continues. With the first stock price being price 0 and the last being price N, there are a total of N+1 prices. There will always be one less return than there are prices. So there will be N returns for N+1 prices. In this case, the returns are

$$r_1 = \frac{S_1}{S_0} - 1,\ r_2 = \frac{S_2}{S_1} - 1,\ r_3 = \frac{S_3}{S_2} - 1, ...,\ r_N = \frac{S_N}{S_{N-1}} - 1$$

The Arithmetic Average Return

Let us calculate the arithmetic average return:

$$\bar{R} = \frac{\sum_{i=t}^{N} r_i}{N}$$

[1]If there are any dividends, we should add them to the numerator. Technically the value of the dividends inserted into the numerator should reflect the payment of the dividends and any interest earned on them from the date of receipt of the dividends until time t. We will assume no dividends in this document.

[2]"Price relative" means "relative price," and indeed one could use that terminology as well.

This calculation is simple. It is nothing more than an average. But the simple average does not tell the full picture. We might want to know at what rate our money grew to get from S_0 to S_N. This rate is the compound return or geometric mean return.

The Geometric Average Return

This measure is found as

$$\bar{G} = \sqrt[N]{\prod_{i=1}^{N}\left(\frac{S_i}{S_{i-1}}\right)} - 1$$

This large "Pi" inside the radical sign is a symbol for "product," meaning to multiply. Thus, we multiply the price relatives. After determining the product of the price relatives, we take the N*th* root.

The geometric mean return formula can, however, be simplified as follows:

$$\bar{G} = \sqrt[N]{\prod_{i=1}^{N}\left(\frac{S_i}{S_{i-1}}\right)} - 1 = \sqrt[N]{\frac{S_1}{S_0}\frac{S_2}{S_1}\frac{S_3}{S_2}\cdots\frac{S_{N-1}}{S_{N-2}}\frac{S_N}{S_{N-1}}} - 1$$

$$= \sqrt[N]{\frac{\cancel{S_1}}{S_0}\frac{\cancel{S_2}}{\cancel{S_1}}\frac{\cancel{S_3}}{\cancel{S_2}}\cdots\frac{\cancel{S_{N-1}}}{\cancel{S_{N-2}}}\frac{S_N}{\cancel{S_{N-1}}}} - 1 = \sqrt[N]{\frac{S_N}{S_0}} - 1$$

In other words, we need only the last price and the first price. The geometric mean simply measures the rate at which the first price grows to get to the last price.

The geometric mean will always be less than the arithmetic mean unless all of the returns are the same.

The Volatility

Finally, we need a measure of volatility that will help us gauge how much movement there is around the mean. Volatility serves as a measure of risk. We typically use the variance or its square root, the standard deviation. The variance is found as follows:

$$\sigma^2 = \frac{\sum_{i=1}^{N}(r_i - \bar{R})^2}{N-1}$$

Note that we divide by N-1. Subtracting 1 from N is an adjustment required when dealing with sample data.

The standard deviation is simply the square root of the variance:

$$\sigma = \sqrt{\sigma^2}$$

An Example (Example 1)

Consider the following sequence of prices:

Time	Price
0	100
1	98
2	101
3	94
4	97
5	102

First let us find the returns:[3]

Time	Price	Price Relative	Return
0	100		
1	98	98/100 = 0.9800	0.9800 – 1 = -0.0200
2	101	101/98 = 1.0306	1.0306 – 1 = 0.0306
3	94	94/101 = 0.9307	0.9307 – 1 = -0.0693
4	97	97/94 = 1.0319	1.0319 - 1 = 0.0319
5	102	102/97 = 1.0515	1.0515 – 1 = 0.0515

Now we can calculate the arithmetic mean:

$$\bar{R} = \frac{-0.0200 + 0.0306 - 0.0693 + 0.0319 + 0.0515}{5} = 0.0049$$

Thus, on average we earned a return of about a positive 0.5%. The geometric mean is

$$\bar{G} = \sqrt[5]{\frac{102}{100}} - 1 = 0.0040$$

The geometric mean indicates that our money grew at a rate of 0.4% per period. That is, if $100 grows in five periods at a rate of 0.0040, it will grow to a value of 102.

The standard deviation is

$$\sigma^2 = \sqrt{\frac{(-0.0200-0.0050)^2 + (0.0306-0.0050)^2 + (-0.0693-0.0050)^2 + (0.0319-0.0050)^2 + (0.0515-0.0050)^2}{4}}$$

$$= 0.0492$$

Some Comparative Examples

In the examples below, I will not show the detailed calculations. You should do them yourself to be sure you can get the right answers. These examples will show different characteristics of the arithmetic and geometric returns. The example above will be called Example 1.

Example 2. Low Variation

Time	Price	Price Relative	Return
0	100		
1	101	101/100 = 1.0100	1.0100 – 1 = -0.0100
2	102	102/101 = 1.0099	1.0099 – 1 = 0.0099
3	100	100/102 = 0.9804	0.9804 – 1 = -0.0196
4	99	99/100 = 0.9900	0.9900 – 1 = -0.0100
5	101	101/99 = 1.0202	1.0202 – 1 = 0.0202

$$\bar{R} = 0.0021$$

$$\bar{G} = 0.0020$$

$$\sigma = 0.0163$$

Notice in comparison to Example 1, that the volatility is much lower. Note also that the geometric mean remains below the arithmetic mean, but the difference is very small.

[3]All of the calculations were done on a spreadsheet, which means that they may differ slightly from manual calculations.

Example 3. High Variation

Time	Price	Price Relative	Return
0	100		
1	110	110/100 = 1.1000	1.1000 – 1 = 0.1000
2	98	98/110 = 0.8909	0.8909 – 1 = -0.1091
3	88	88/98 = 0.8980	0.8980– 1 = -0.1020
4	102	102/88 = 1.1591	1.1591– 1 = 0.1591
5	120	120/102 = 1.1765	1.1765 – 1 = 0.1765

$$\bar{R} = 0.0449$$

$$\bar{G} = 0.0371$$

$$\sigma = 0.1403$$

Now notice in comparison to Example 1, that the volatility is much higher. Note also that the geometric mean remains below the arithmetic mean, and the difference is somewhat larger. Is it a general rule that the greater the volatility, the greater the difference between the arithmetic and geometric means.

Example 4. No variation

In this problem all of the returns are the same.[4]

Time	Price	Price Relative	Return
0	100		
1	104	104/100 = 1.04	1.04 – 1 = 0.04
2	108.16	108.16/104 =1.04	1.04 – 1 = 0.04
3	112.49	112.49/108.16 = 1.04	1.04 – 1 = 0.04
4	116.99	116.99/112.49 = 1.04	1.04 – 1 = 0.04
5	121.67	121.67/116.99 = 1.04	1.04 – 1 = 0.04

$$\bar{R} = 0.0400$$

$$\bar{G} = 0.0400$$

$$\sigma = 0.0000$$

Notice that with no variation, the arithmetic and geometric returns are the same.

Example 5. Bankruptcy

In this problem, we assume that the company goes bankrupt, so that the final stock price is zero.

Time	Price	Price Relative	Return
0	100		

[4]I constructed these prices so that each would be 4% higher than the previous price. Therefore, the prices are rounded off and have more significant digits than are shown.

1	200	200/100 = 2.0000	2.0000 – 1 = 1.0000
2	250	250/200 = 1.2500	1.2500 – 1 = 0.2500
3	275	275/250 = 1.1000	1.1000 – 1 = 0.1000
4	200	200/275 = 0.7273	0.7273 – 1 = -0.2727
5	0	0/200 = 0.0000	0 – 1 = -1.0000

$$\bar{R} = 0.0155$$

$$\bar{G} = -1.0000$$

$$\sigma = 0.7324$$

Note that an investment of $100 in this stock results in a complete loss of all of the investor's money by period 5. But the average return is a positive 1.55%! This example indicates that the arithmetic and geometric returns both provide information. Which one is preferable depends on the context. In estimating expected returns for investment decision making purposes, we prefer the arithmetic mean. It is a better reflection of the return one would expect, on average, which is what the expected return indicates. But the geometric mean return is a better indicator of performance and is clearly influenced by the risk. In fact, when stock returns are normally distributed there is an approximate relationship between the arithmetic and geometric mean returns that is related to the risk.

The Arithmetic and Geometric Mean Returns in a Normal Distribution

When returns follow the normal distribution, the arithmetic and geometric mean returns are related in the following approximate manner:

$$\bar{G} \cong \bar{R} - \frac{\sigma^2}{2}$$

Note that the more risk (greater σ^2), the lower is the geometric mean. This formula will often work quite well even if returns are not normally distributed. In the examples above, which are not normally distributed, we get the following results:

Example	$\bar{R}$	σ	Approx. $\bar{G}$	Actual $\bar{G}$
1	0.0050	0.0492	0.0037	0.0040
2	0.0021	0.0163	0.0020	0.0020
3	0.0449	0.1403	0.0351	0.0350
4	0.0400	0.0000	0.0400	0.0400
5	0.0155	0.7324	-0.2527	-0.2527

The approximations are pretty good in every case.

Continuously Compounded Returns

Sometimes it is helpful to express returns as continuously compounded returns. For example, suppose a stock starts at 100 and goes to 110 in one year. The simple return is 110/100 – 1 = 0.10, or 10%. The simple returns assumes that value compounds only once per period. Continuous compounding assumes that the return grows on itself at a continuous rate. In between simple and continuous returns are returns that are compounded semi-annually, quarterly, monthly, daily, etc.

Assume $1 is invested for m years. Let q be the number of compounding periods per year. With monthly compounding for two years, m = 2 and q = 4. Thus, there are 8 compounding periods over the two years. In general, the future value of $1 invested for m periods at the rate r with compounding q times per year is:

$$\left(1+\frac{r}{q}\right)^{mq}$$

When q is infinite, the above expression must be evaluated by taking limits. We skip the math but simply note that the expression becomes

$$e^{rm}$$

In other words, we use "e", the base of the natural log system and raise it to the r*m power. As an example, $1 compounded at 5% for two years grows as follows:

Compounding period	Future value
Annually	$(1.05)^{1*2} = 1.1025$
Semiannually	$\left(1+\frac{.05}{2}\right)^{2*2} = 1.1038$
Quarterly	$\left(1+\frac{.05}{4}\right)^{2*4} = 1.1045$
Monthly	$\left(1+\frac{.05}{12}\right)^{2*12} = 1.1049$
Weekly	$\left(1+\frac{.05}{52}\right)^{2*52} = 1.1051$
Daily	$\left(1+\frac{.05}{365}\right)^{2*365} = 1.1052$
Continuously	$e^{.05*2} = 1.1052$

Note that there is no discernible difference (to four decimal places) between daily and continuous compounding.

Turning the problem around, we might ask, "What rate of return allows a current value to grow to another value?" Let us say that $100 grows to $110 in one year. What is the rate? With annual compounding, the rate is simply 110/100 – 1 = 0.10. We call this the simple rate. If the price of $100 grows continuously to get to $110, the rate of growth will be less. We get this rate by taking the natural log of one plus the simple rate. That is, the continuous rate here is

$$\ln(1.10) = 0.0953$$

Thus, the rate is, 9.53%. Similarly, $100 invested at 9.53% compounded continuously for one year grows to $110:

$$\$100e^{0.10*1} = \$110$$

(subject to a round-off error.)

Now reconsider the prices and returns in Example 1.

Time	Price	Price Relative	Return
0	100		
1	98	98/100 = 0.9800	0.9800 – 1 = -0.0200
2	101	101/98 = 1.0306	1.0306 – 1 = 0.0306
3	94	94/101 = 0.9307	0.9307 – 1 = -0.0693
4	97	97/94 = 1.0319	1.0319 – 1 = 0.0319
5	102	102/97 = 1.0515	1.0515 – 1 = 0.0515

The continuously compounded returns are:

$$\ln(0.9800) = -0.0202$$
$$\ln(1.0306) = 0.0301$$
$$\ln(0.9307) = -0.0718$$
$$\ln(1.0319) = 0.0314$$
$$\ln(1.0515) = 0.0502$$

Notice that the continuously compounded returns are a little less than the corresponding simple returns. The difference is small when the simple return is small and gets larger when the simple return is larger. For example, a simple return of 25% is a continuously compounded return of ln(1.25) = 0.2231.

The continuously compounded return is related to the geometric mean return. Recall that the formula for the geometric mean return is

$$\bar{G} = \sqrt[N]{\prod_{i=1}^{N}\left(\frac{S_i}{S_{i-1}}\right)} - 1 = \sqrt[N]{\frac{S_1}{S_0}\frac{S_2}{S_1}\frac{S_3}{S_2}\cdots\frac{S_{N-1}}{S_{N-2}}\frac{S_N}{S_{N-1}}} - 1$$

Take the natural log of one plus the geometric mean:

$$\ln(1+\bar{G}) = \ln\left(\sqrt[N]{\prod_{i=1}^{N}\left(\frac{S_i}{S_{i-1}}\right)}\right)$$
$$= \frac{1}{N}\ln\left(\frac{S_1}{S_0}\frac{S_2}{S_1}\frac{S_3}{S_2}\cdots\frac{S_{N-1}}{S_{N-2}}\frac{S_N}{S_{N-1}}\right)$$
$$= \frac{\ln\left(\frac{S_1}{S_0}\right) + \ln\left(\frac{S_2}{S_1}\right) + \ln\left(\frac{S_3}{S_2}\right) + \ldots + \left(\frac{S_{N-1}}{S_{N-2}}\right) + \ln\left(\frac{S_N}{S_{N-1}}\right)}{N}$$

Thus, the natural log of the geometric mean return is the average continuously compounded return. In Example 1,

$$\frac{-0.0202 + 0.0301 - 0.0718 + 0.0314 + 0.0502}{5} = 0.0040$$
$$\ln(1.0040) = 1.0040$$

(there is some rounding error)