

MBA Teaching Note 07-03
Derivations of Present Value Formulas

Finance students learn several key present value formulas. For the most part they can learn what they need to know in introductory classes without knowing where the formulas come from. But some students are curious and others do not believe in a formula until they know where it comes from. This note shows how the major present value formulas are derived.

Present Value of an Annuity

Consider an annuity of N payments of C dollars each. We know that the present value is

$$\begin{aligned} V_0 &= \frac{C}{(1+r)^1} + \frac{C}{(1+r)^2} + \dots + \frac{C}{(1+r)^N} \\ &= C \left(\frac{1}{(1+r)^1} + \frac{1}{(1+r)^2} + \dots + \frac{1}{(1+r)^N} \right) \\ &= C \sum_{i=1}^N \frac{1}{(1+r)^i}. \end{aligned}$$

We are told that this formula simplifies to

$$V_0 = C \left(\frac{1 - (1+r)^{-N}}{r} \right).$$

Now let us see how this formula is obtained. Rewrite the above equation and number it:

$$V_0 = C \left(\frac{1}{(1+r)^1} + \frac{1}{(1+r)^2} + \dots + \frac{1}{(1+r)^{N-1}} + \frac{1}{(1+r)^N} \right) \quad (1)$$

Divide the left- and right-hand sides by $1+r$:

$$\frac{V_0}{1+r} = C \left(\frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \dots + \frac{1}{(1+r)^N} + \frac{1}{(1+r)^{N+1}} \right). \quad (2)$$

Subtract equation (2) from equation (1):

$$V_0 - \frac{V_0}{1+r} = C \left(\frac{1}{1+r} - \frac{1}{(1+r)^{N+1}} \right).$$

Solve this equation for V_0 :

$$V_0 \left(1 - \frac{1}{1+r} \right) = C \left(\frac{1}{1+r} - \frac{1}{(1+r)^{N+1}} \right)$$

$$V_0 \frac{1+r-1}{1+r} = C \left(\frac{1}{1+r} - \frac{1}{(1+r)^{N+1}} \right)$$

$$V_0 \left(\frac{r}{1+r} \right) = C \left(\frac{1}{1+r} - \frac{1}{(1+r)^{N+1}} \right)$$

(Multiply by $(1+r)$)

$$V_0 r = C \left(1 - \frac{1}{(1+r)^N} \right)$$

$$V_0 = C \left(\frac{1 - (1+r)^{-N}}{r} \right).$$

Of course, this formula is the discount factor for an annuity.

For an annuity due, meaning that there are N payments but the first starts at time 0, we can view it as an annuity of $N-1$ payments and an immediate lump sum payment at time 0. Thus, we can write its value as

$$V_0 = C + C \left(\frac{1 - (1+r)^{-(N-1)}}{r} \right)$$

For example, suppose there is one payment at time 0, one at time 1, one at time 2, and one at time 3. Drop the first payment for the moment. What remains is an ordinary annuity of three payments. Then add the lump sum at time 0, and you have the above formula. Alternatively, you can get the present value of an annuity due of N periods by getting the present value of an ordinary annuity of N payments and multiplying by $(1+r)$, as follows:

$$\begin{aligned} V_0 &= C \left(\frac{1 - (1+r)^{-N}}{r} \right) (1+r) \\ &= \frac{C(1+r) - C(1+r)^{-(N-1)}}{r} = \frac{C + Cr - C(1+r)^{-(N-1)}}{r} \\ &= \frac{Cr}{r} + C \left(\frac{1 - C(1+r)^{-(N-1)}}{r} \right) \\ &= C + C \left(\frac{1 - (1+r)^{-(N-1)}}{r} \right) \end{aligned}$$

And this is the same formula shown above for the present value of an annuity due.

A perpetuity is a special case of an annuity with an infinite number of payments. Using the discount formula for an ordinary from above, we simply let N approach infinity. In that case $(1+r)^{-\infty}$ will approach zero, leaving,

$$V_0 = \frac{1}{r}.$$

Compound Value of an Annuity

This formula can be easily derived from the present value formula for an ordinary annuity that was presented just above. In this case we want the value at time N of that same stream of payments. We can easily get this formula by multiplying that formula by $(1+r)^N$:

$$\begin{aligned} V_N &= V_0 (1+r)^N \\ &= C \left(\frac{1 - (1+r)^{-N}}{r} \right) (1+r)^N \\ &= C \left(\frac{(1+r)^N - 1}{r} \right) \end{aligned}$$

Now consider a compound annuity due. Suppose there are N payments, the first starting right now. This means there are $N-1$ payments of an ordinary annuity and one payment occurring today. We typically want to know the value of this annuity at the time of the last payment, time $N-1$. This statement may seem confusing: there are N payments, but we want to know the value at time $N-1$. The reason for saying it this way is that we usually want to know the value at the time of the later payment. If we separate the payments into one payment now and $N-1$ payments starting one period later, we want to know the annuity value at the time of payment $N-1$.

So we can treat this problem as ordinary compound annuity of $N-1$ payments and a lump sum made today and compounded for $N-1$ periods.

$$V_{N-1} = C(1+r)^{N-1} + C\left(\frac{(1+r)^{N-1} - 1}{r}\right)$$

This formula can be rewritten as

$$\begin{aligned} V_{N-1} &= C(1+r)^{N-1} + C\left(\frac{(1+r)^{N-1} - 1}{r}\right) \\ &= C(1+r)^{N-1}\left(1 + \frac{1}{r}\right) - Cr = C(1+r)^{N-1}\left(\frac{1+r}{r}\right) - Cr \\ &= C\left(\frac{(1+r)^N}{r} - \frac{1}{r}\right) \\ &= C\left(\frac{(1+r)^N - 1}{r}\right) \end{aligned}$$

And this is the formula we gave above for an N -period ordinary annuity. Indeed, a compound ordinary annuity of $N-1$ periods is the same as an ordinary annuity of N periods. This statement is true because the first deposit compounds for N periods, the second for $N-1$, the third for $N-2$, and so on until the last compounds for zero periods. This result is exactly what happens when the payments start one period earlier and there are N payments in total. As another way to look at it, suppose one year ago you made a decision to save C dollars each year for four years with the first payment starting one period later. You can easily find its compound value at time four as a four-period ordinary annuity. Now, roll forward one year, and today is one period after you made that decision. You are making your first payment right now. What is the compound value three years from now? It is the same as it was one year ago.

Constant Growth Model

This model is widely used to find the price of a stock. Starting at time 0, we let the next cash flow be specified as C_1 . Each succeeding dividend is higher than the preceding one by a factor $(1+g)$. This growth continues forever. The present value is

$$V_0 = \frac{C_1}{1+r} + \frac{C_1(1+g)}{(1+r)^2} + \frac{C_1(1+g)^2}{(1+r)^3} + \dots + \frac{C_1(1+g)^{\infty-1}}{(1+r)^\infty} + \frac{C_1(1+g)^\infty}{(1+r)^{\infty+1}}. \quad (3)$$

Now let us multiply this equation by $(1+g)/(1+r)$:

$$V_0\left(\frac{1+g}{1+r}\right) = \frac{C_1(1+g)}{(1+r)^2} + \frac{C_1(1+g)^2}{(1+r)^3} + \frac{C_1(1+g)^3}{(1+r)^4} + \dots + \frac{C_1(1+g)^\infty}{(1+r)^{\infty+1}} + \frac{C_1(1+g)^{\infty+1}}{(1+r)^{\infty+2}}. \quad (4)$$

Now subtract equation (4) from equation (3):

$$\begin{aligned} V_0 - V_0\left(\frac{1+g}{1+r}\right) &= \frac{C_1}{1+r} + \frac{C_1(1+g)}{(1+r)^2} + \frac{C_1(1+g)^2}{(1+r)^3} + \dots + \frac{C_1(1+g)^{\infty-1}}{(1+r)^\infty} + \frac{C_1(1+g)^\infty}{(1+r)^{\infty+1}} \\ &\quad - \left(\frac{C_1(1+g)}{(1+r)^2} + \frac{C_1(1+g)^2}{(1+r)^3} + \frac{C_1(1+g)^3}{(1+r)^4} + \dots + \frac{C_1(1+g)^\infty}{(1+r)^{\infty+1}} + \frac{C_1(1+g)^{\infty+1}}{(1+r)^{\infty+2}} \right) \\ V_0\left(1 - \frac{1+g}{1+r}\right) &= \frac{C_1}{1+r} - \frac{C_1(1+g)^{\infty+1}}{(1+r)^{\infty+2}}. \end{aligned}$$

At this point we make the assumption that the growth rate is less than the discount rate. If that is the case, then the second term on the right-hand side is equal to zero, leaving¹

$$V_0 \left(\frac{1+r-(1+g)}{1+r} \right) = \frac{C_1}{1+r}$$

$$V_0 \left(\frac{r-g}{1+r} \right) = \frac{C_1}{1+r}$$

$$V_0 = \frac{C_1}{r-g}.$$

Of course, this is the formula we are interested in. Typically, the cash flow is a dividend (replace C_1 with D_1) and sometimes the discount rate is expressed as k or another symbol. The present value is usually replaced by the price symbol, P_0 .

The perpetuity is a case of zero growth. Thus, let $g = 0$ and the above formula simplifies to the well-known perpetuity formula as shown above.

Constant Growth for a Finite Period of Time

In the above formula, the constant growth continues forever. In some problems, constant growth occurs for only a finite period of time, such as the following.

$$V_0 = \frac{C_1}{1+r} + \frac{C_1(1+g)}{(1+r)^2} + \frac{C_1(1+g)^2}{(1+r)^3} + \dots + \frac{C_1(1+g)^{N-2}}{(1+r)^{N-1}} + \frac{C_1(1+g)^{N-1}}{(1+r)^N}. \quad (5)$$

Note in this problem that there are N cash flows, where the N^{th} cash flow is higher than the first by the factor $(1+g)^{N-1}$. Now, multiply equation (5) by $(1+g)/(1+r)$:

$$V_0 \left(\frac{1+g}{1+r} \right) = \frac{C_1(1+g)}{(1+r)^2} + \frac{C_1(1+g)^2}{(1+r)^3} + \frac{C_1(1+g)^3}{(1+r)^4} + \dots + \frac{C_1(1+g)^{N-1}}{(1+r)^N} + \frac{C_1(1+g)^N}{(1+r)^{N+1}}. \quad (6)$$

Subtract equation (6) from equation (5):

$$V_0 - V_0 \left(\frac{1+g}{1+r} \right) = \frac{C_1}{1+r} + \frac{C_1(1+g)}{(1+r)^2} + \frac{C_1(1+g)^2}{(1+r)^3} + \dots + \frac{C_1(1+g)^{N-1}}{(1+r)^N}$$

$$- \left(\frac{C_1(1+g)}{(1+r)^2} + \frac{C_1(1+g)^2}{(1+r)^3} + \frac{C_1(1+g)^3}{(1+r)^4} + \dots + \frac{C_1(1+g)^{N-1}}{(1+r)^N} + \frac{C_1(1+g)^N}{(1+r)^{N+1}} \right)$$

$$V_0 \left(1 - \frac{1+g}{1+r} \right) = \frac{C_1}{1+r} - \frac{C_1(1+g)^N}{(1+r)^{N+1}}.$$

Now, solve for V_0 :

¹The second term on the right-hand side involves raising a factor $(1+g)$ to the power $\infty - 1$ divided by a larger factor, $(1+r)$, raised to a higher power, ∞ . When the power of the denominator gets large enough, the denominator swamps the numerator, resulting in a value of zero for that term.

$$V_0 \left(\frac{1+r-(1+g)}{1+r} \right) = C_1 \left(\frac{1}{1+r} - \frac{(1+g)^N}{(1+r)^{N+1}} \right)$$

$$V_0 \left(\frac{r-g}{1+r} \right) = C_1 \left(\frac{1}{1+r} - \frac{(1+g)^N}{(1+r)^{N+1}} \right)$$

$$V_0(r-g) = C_1 \left(1 - \frac{(1+g)^N}{(1+r)^N} \right)$$

$$V_0 = C_1 \frac{\left(1 - \left(\frac{1+g}{1+r} \right)^N \right)}{r-g}.$$

This formula is not specifically covered in my class, but it can be used when there is constant growth for a finite period of time. Note that if N is allowed to be infinite and provided $g < r$, then the expression $\left(\frac{1+g}{1+r} \right)^\infty$ goes to zero and the formula reduces to the same one we previously covered for the case of constant growth at $g < r$ forever.