

Tree Pricing of Bonds and Interest Rate Derivatives: III

Essay 57

This essay is the third in a series on the pricing of bonds and interest rate derivatives using binomial trees. In the previous essay we priced bonds and derivatives directly on the bonds. In this essay, we price derivatives directly on the one-period interest rate implied by the tree we developed in Essay 54. You may wish to have a copy of the tree in Essay 55 handy as I will be referring to it often.

An interest rate derivative is one in which the payoff is based on an interest rate. Swaps, options, and forward rate agreements (FRAs) are the most common examples. In the real world, the interest rate might be 90 day LIBOR, but that is by no means the only rate used in such contracts. In this essay we shall simply use the one-period rate as the underlying, and we shall do only FRAs and interest rate caps and floors. We shall cover swaps and swaptions in the next two essays.

In pricing interest rate derivatives we must first use the tree of zero coupon bond prices to develop a tree of the evolution of one-period rates. These will be found by inferring the interest rate from the price of the various one dollar discount bonds whose prices are given in the tree and are denoted as $P(0,1)$, $P(1,2)$, $P(2,3)$, $P(3,4)$, and $P(4,5)$. Note that this is not the price of a single instrument as it evolves through time but rather the price of a zero coupon that matures

one period later. This will always be a two-period bond the previous period, which then evolves into a one-period bond. In our example, $P(0,1) = 0.905$ so the one-period rate is $1/P(0,1) - 1 = 1/0.905 - 1 = 0.105$ or 10.5%. We call this $r(0,1)$. The next period, the one-period bond price can be $P(1,2) = 0.893$ or 0.920. This means that the one-period rate will be either 12% or 8.74%. (Note that the bond prices are rounded off; greater precision is in the interest rates.) At period 2 there are three possible one-period rates: 13.47%, 10.17%, or 6.96%. Summarizing the entire process, we obtain the tree of one-period rates below:

Binomial Evolution of the One-Period Rate

		Time 3		Time 4
$r(0,1) = 10.50$	$r(1,2) = 12.00$	$r(2,3) = 13.47$	$r(3,4) = 14.95$	$r(4,5) = 16.45$
		$r(2,3) = 10.17$	$r(3,4) = 11.60$	$r(4,5) = 13.66$
	$r(1,2) = 8.74$	$r(2,3) = 10.17$	$r(3,4) = 8.35$	$r(4,5) = 9.76$
		$r(2,3) = 6.96$	$r(3,4) = 5.20$	$r(4,5) = 6.57$
				$r(4,5) = 3.46$

Note: These rates may not exactly equal those that you would obtain manually. They were obtained using a spreadsheet, which uses more exact values of the spot and forward prices.

Of course, there is no reason why we must focus solely on the one-period rate, but the tree for any other rate would be obtained in a similar manner, and the derivative pricing procedure to follow would be the same.

Let us first price the forward rate agreement or FRA, which we introduced in Essay 6. Standard FRAs mature at a given point in time and pay off at that point. They call for one party to make an interest payment at whatever rate occurs at that time. In the tree above consider an FRA expiring at time 2. In the real world, this might correspond to entering into a six-month FRA on 90 day LIBOR. Since the value of an FRA at Time 0 is zero, the fixed rate, set at Time 0, is the rate such that the risk neutral expected payoff is zero. The three

possible one-period rates at Time 2 are 13.47, 10.17, and 6.96. The risk neutral probabilities of these rates occurring are given by the binomial probabilities $(0.52)^2 = 0.2704$, $2(0.52)(0.48) = 0.4992$, and $(0.48)^2 = 0.2304$. Thus the rate on a two-period FRA, which we denote as $FRA(0,2)$, is the rate such that $0.2704(13.47 - FRA(0,2)) + 0.4992(10.17 - FRA(0,2)) + 0.2304(6.96 - FRA(0,2)) = 0$. Solving this simple equation gives $FRA(0,2) = 10.32$. Thus, if a firm entered into a two-period FRA on the one-period rate, it would agree to make an interest payment at Time 2 of 10.32% and agree to receive an interest payment at that date of whatever the one-period rate is. Other rates for various FRAs are $FRA(0,1) = 10.43$, $FRA(0,3) = 10.20$, and $FRA(0,4) = 10.07$. Of course the actual payoffs would be determined by applying these rates to a notional principal and prorating over the appropriate number of days that the rate corresponds to. We will not concern ourselves here with these matters as they are straightforward. We shall quote all results in terms of a rate only.

Now let us turn to interest rate caps and floors, which are options on interest rates and were introduced in Essay 6. A cap (floor) is like a call (put) and pays off if the interest rate is greater (less) than the strike rate at expiration or earlier if American-style and exercised early. The typical arrangement is that the actual payment occurs one period after the expiration. This corresponds to the convention found on floating rate loans, where the rate is set at the beginning of the period and the actual interest payment occurs at the end of the period. Caps and floors are actually a series of independent options called caplets and floorlets. The expiration of each caplet or floorlet is usually set to correspond to the reset dates on floating rate securities held or issued by the purchaser. The decision to exercise a caplet or floorlet is independent of the decision to exercise any other caplet or floorlet. That is, exercising one caplet or floorlet does not preclude or require exercise of any other.

The value of a cap or floor is the sum of the values of each component caplet or floorlet. Thus, we price the cap or floor as a portfolio of options. Consider a four-period cap on the one-period rate. The cap consists of four component caplets, one expiring at Time 1, another at Time 2, another at Time 3, and another at Time 4. On each caplet expiration date, one caplet expires and is in-the-money (out-of-

the-money) if the one-period rate is higher (lower) than the strike rate chosen by the buyer. The actual payoff is made one period later, so its value must be discounted back at the appropriate one-period rate.

Let our cap have a strike of 9%. The five possible one-period rates at Time 4 are 16.45%, 13.06%, 9.76%, 6.57%, and 3.46%. In the uppermost state, the rate of 16.45% means that the final caplet expires in-the-money for a payoff of $16.45\% - 9\% = 7.45\%$. This payment will be made one period later, so we discount it using the one-period discount factor at that node in the tree, which is 0.859. Thus, the value of the option at expiration when the one-period rate is 16.45% is $0.0745(0.859) = 0.0640$, which is interpreted as 6.40%. Moving down the tree at Time 4 to the next four nodes would produce caplet values of 0.0359, 0.0070, 0.0000, and 0.0000, reflecting the rates of 13.06%, 9.76%, 6.57%, and 3.46% and one-period discount factors of 0.885, 0.911, 0.938, and 0.967.

Remember that we are still just finding the price of a single caplet, the interest rate call option that expires at Time 4. We have five possible values as noted above. At each node, we step back one unit of time and find the probability-weighted discounted value. For example, in the Time 3 topmost node, the caplet would be worth 6.40% if the rate goes up at Time 4 or 3.59% if the rate goes down at Time 4. Using our probabilities of 0.52 and 0.48, we have an expected rate of $0.52(6.40) + 0.48(3.59) = 5.05$. This value is then discounted by the one-period factor at the topmost node of Time 3 of 0.870 to give a value of 4.39 in that node.

As usual, we proceed backwards through the tree, successively discounting the probability-weighted average of the next two possible values. At Time 0 we have the value of the caplet that expires at Time 4. But this is just the value of one of four caplets that make up this cap. We also have caplets expiring at Times 1, 2, and 3. Repeating this procedure to value those options, we obtain the full value of the cap, i.e., the sum of the values of the four caplets, as 4.93%. To value a floor, we follow the same procedure but have the floor pay off like an interest put, i.e., have a positive value at expiration if the one-period rate is below the strike rate.

Because they are so often used to manage the risk of resetting the interest rate on a floating rate loan or security, which is done on a

specific date, caps and floors are typically structured as European-style, but they can be and occasionally are done as American-style. If so, the exercise value of each option at each node prior to expiration would replace the computed value if the former were higher.

In the next essay, the fourth in this series on pricing interest rate derivatives, we address the pricing of swaps and swaptions.

For More Reading

Jarrow, Robert. *Modelling Fixed Income Securities and Interest Rate Derivatives*. New York: McGraw-Hill (1996).