ESSAY 31

Risk-Neutral Pricing of Derivatives: I

This essay on the topic of risk-neutral pricing is the first of two essays that address this important topic. It is undoubtedly one of the most critical, most misunderstood, and most misapplied concepts in finance. It turns out that derivatives can be priced by assuming investors are risk neutral, which sounds pretty radical given that no one is actually risk neutral. Yet how a person feels about risk is essentially irrelevant where derivatives are concerned.

We start by reviewing what we mean by risk preferences. Suppose you are offered an opportunity to engage in a coin-tossing game. If the coin turns up heads, you win $1. If it turns up tails, you win nothing and lose nothing. Would you play this game? The answer depends on whether it costs to play the game. Let us first say it costs nothing. Then of course you would play. You have nothing to lose. You have exhibited the first axiom of utility, which is *nonsatiation*. Given the opportunity to increase your wealth without regard to any risk, you would do so.

Suppose now that you are charged a fee to play the game. Let us say the fee is the expected outcome of $0.50. Do you play? You recognize that over the long run, you would win only what you paid, which would not be acceptable to a risk-averse investor. The feelings of disappointment, frustration, and fear of further losses would mean that an expected return equivalent to the expected payoff would not compensate you for the risk incurred. Suppose you played twice, losing first and then winning. While you would have broken even, you would prefer to have not played at all. Risk-averse people would, therefore, pay less than the expected outcome to play the game. The difference between what you pay and the expected outcome is the *risk premium*, the additional expected return that compensates you for the taking on the risk.

If you were risk neutral, however, you would willingly pay $0.50 to play the game. Your risk neutrality would mean that every loss you incurred would not bother you; you would see it as only a sidestep on the path to possible future gains.

We typically feel that most people are risk averse, but we do observe people who appear to be risk neutral and maybe even risk loving. The latter would pay more than \$0.50 to play. They enjoy the risk and would willingly pay to assume more of it. Gamblers may appear to be anywhere from risk neutral to risk loving, but in fact they are probably as risk averse as anyone else. They receive a nonpecuniary reward in the form of the thrill of playing. After accounting for that reward, they are probably risk averse. Most of modern investment theory assumes that investors are risk averse. Hence, the price one pays for a security is set so that the investor expects a minimum return of the risk-free rate and a risk premium commensurate with the degree of risk, however measured. This statement does not, of course, mean that after the fact the investor would earn a risk premium. It means only that the risk premium is expected before the fact. In the long run, however, investors must receive risk premiums after the fact or they would stop investing in risky ventures.

Suppose an investor purchases an asset and holds it until a future date. Recognizing the risk, the investor decides to hedge the position with a forward (or futures) contract that calls for her to deliver the asset at the specific future date. What is an appropriate price for the asset and the futures? The asset will be priced, as described, such that the investor expects to earn a risk-free return and a risk premium. The existence and use of the derivative contract do not affect the underlying asset's price.[1]

The derivative price is found by invoking the no-arbitrage principle. If the investor buys the asset at S and immediately sells it with a forward contract, she guarantees receipt of a price known today, which we shall label as F. The investor might incur storage costs, but we will assume none here for simplicity. We shall assume the asset offers no yield like a dividend though this could be easily added. Let r be the risk-free rate. The transaction guarantees that the investor will have F dollars at the end of the holding period. She will, however, have paid S dollars at the start, which effectively becomes $S(1 + r)$ by the end of the transaction, reflecting the opportunity cost. The transaction is risk free and thus should not be expected to produce a risk premium. In addition, the transaction should generate no profit beyond the interest. Otherwise it would mean that a risk-free transaction would earn

[1] We could argue the point of the tail wagging the dog—meaning that maybe the derivative market is driving the underlying market instead of vice versa—but we shall not do so here.

more than the risk-free rate. Such a transaction would be very attractive, and everyone would want to do it. To rule out this opportunity for profitable arbitrage, the forward price F must equal the spot price grossed up by the factor $1 + r$.

Now, how does risk-neutral pricing come into play? Suppose I told you that the investor was not risk averse, but rather was risk neutral or even risk loving but I gave you the spot price S. Without knowing where that price came from, you would still arrive at the same forward price formula, $F = S(1 + \mathrm{r})$. The spot price S is determined independently of the forward price. It reflects the investor's risk preferences, the risk-free rate, and the expected future price of the underlying. So regardless of how the investor feels about risk, if given the spot price, the forward price is still the spot price—however obtained—factored up by the risk-free rate.[2]

This result suggests that the risk preferences of investors do not matter when arriving at the derivative price. That being the case, we could just as easily assume that investors are risk neutral. If that were true, then risk premiums do not exist so every asset has an expected return equal to the risk-free rate. We would be able to find the expected future spot price by grossing it up by the risk-free rate and that would equal the forward price.

Risk-neutral pricing is typically explained in the context of an option and often with the binomial model. I am taking a different and, I hope, simpler approach. Options are more complicated than forwards because it is necessary to look at two distinct states of nature at expiration, in-the-money and out-of-the-money. Forwards (and futures) can be evaluated by considering only one state or outcome: The spot price is at any given level, and the contract pays off the spot price minus the original forward price. That one outcome subsumes all possible outcomes.

Let us look at the problem another way. Suppose we find an identical planet to ours in which the populace is risk neutral. Let us call it Planet RN. Planet RN is as advanced as ours with similar financial markets and securities. We find an identical asset trading on Earth and Planet RN with the same probability distribution. That asset will trade at a lower price on Earth than on Planet RN because earthlings are not risk neutral.

Does everything I have said about risk neutrality imply that forwards will trade at the same price on both planets? Recall that I stated that investor risk preferences are irrelevant to the pricing of derivatives. That statement sounds like the forward price would be the same in both cases, but

[2] Of course, any cash flows, storage costs, or convenience yield would effect the forward price, as we have previously covered.

it is not. Investor risk preferences are not irrelevant to the *absolute* price of the derivative. They are irrelevant to the *relative* price of the derivative. By relative price, we mean the price of the derivative in relationship to the price of the underlying asset. In other words, the mathematical relationship between the derivative and its underlying asset is invariant to the risk preferences of investors who hold the asset. That means that the underlying asset price is taken as given, having been determined from the risk preferences of investors. Given that price, the derivative price is determined according to the cost-of-carry formula or whatever other formula is appropriate.

This little trick of using risk neutrality has made it possible to solve many complex derivative pricing problems, but it has been misinterpreted and misstated in many cases. Using risk neutrality is not the same as assuming that investors are risk neutral. Moreover, using risk neutrality to derive a pricing model does not mean that absolute derivative prices are invariant to risk preferences. In fact, absolute derivative prices are quite sensitive to risk preferences, but this sensitivity manifests in the form of the derivative's sensitivity to the price of the underlying asset, which is determined by risk preferences. Relative derivative prices are invariant to risk preferences. This distinction is rarely made but doing so helps clear up the confusion engendered when so many financial engineers imply that they are assuming risk neutrality to derive a pricing model. They are not really *assuming* risk neutrality. They are simply *using* risk neutrality.

FOR MORE READING

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TEST YOUR KNOWLEDGE

1. Explain the difference between a person who is risk averse and one who is risk neutral.
2. Why does the standard approach to derivative pricing result in the appearance of the expected return equaling the risk-free rate?
3. Explain why a risk-neutral investor and a risk-averse investor would arrive at the same derivative price.